Motion and transport in a near-spherical shell
Geometric approximations

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Transport in Cartesian coordinates

Consider

- a fluid parcel with position $\mathbf{x}(t)$ and velocity $\dot{\mathbf{x}}(t)$
- carrying a property such as specific (= per unit mass) entropy or specific salinity
- which evolves according to

$$\dot{s} = \frac{q}{T}$$

Now

- fluid parcels are everywhere
- Hence we can regard $\dot{\mathbf{x}}$ and s as fields $\dot{\mathbf{x}}(x, y, z, t), s(x, y, z, t)$
- The above time derivative is the *material* or *Lagrangian* derivative
- By the chain rule :

$$\dot{s} = \frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \dot{x}\frac{\partial s}{\partial x} + \dot{y}\frac{\partial s}{\partial y} + \dot{z}\frac{\partial s}{\partial z}$$

Advective form

Mass budget in Cartesian coordinates

Consider

- a mixture of molecules O2, N2 ...
- each with number per unit volume $N_n(\mathbf{x}, t)$
- And moving with velocity $\dot{\mathbf{x}}_n(\mathbf{x}, t)$
- Then

$$\frac{\partial N_n}{\partial t} + \mathrm{div}\,(N_n \dot{\mathbf{x}}_n) = 0$$

Now

- Assume all species have the same (macroscopic) velocity
- Define density as :

$$\rho \equiv \sum_n M_n N_n$$

- Then

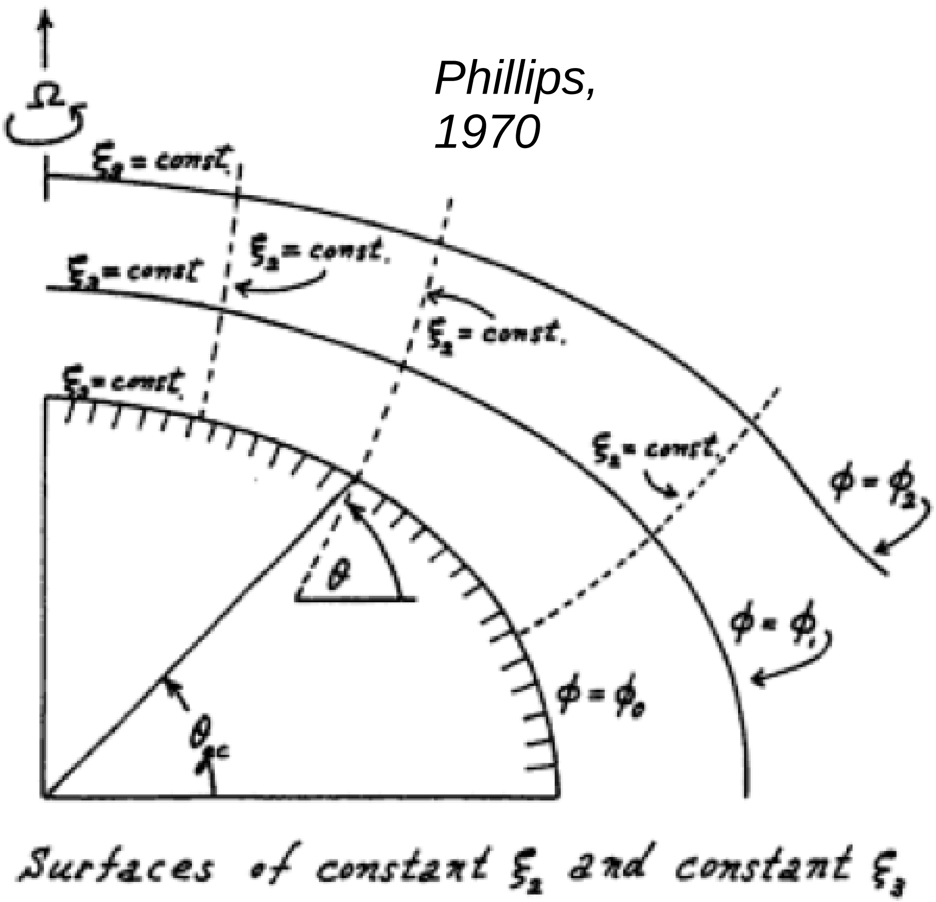
$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \dot{\mathbf{x}}) = 0$$

- With some algebra :

$$\frac{\partial \rho s}{\partial t} + \mathrm{div}\,(\rho s \dot{\mathbf{x}}) = \rho \frac{q}{T}$$

Flux form
= conservative form

Transport in curvilinear coordinates



Phillips, 1970

The geometry of the Earth is better suited to *curvilinear* coordinates

- For instance spherical coordinates λ, ϕ, r
- In general : ξ^1, ξ^2, ξ^3
- By the chain rule :

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + u^1 \frac{\partial s}{\partial \xi^1} + u^2 \frac{\partial s}{\partial \xi^2} + u^3 \frac{\partial s}{\partial \xi^3}$$

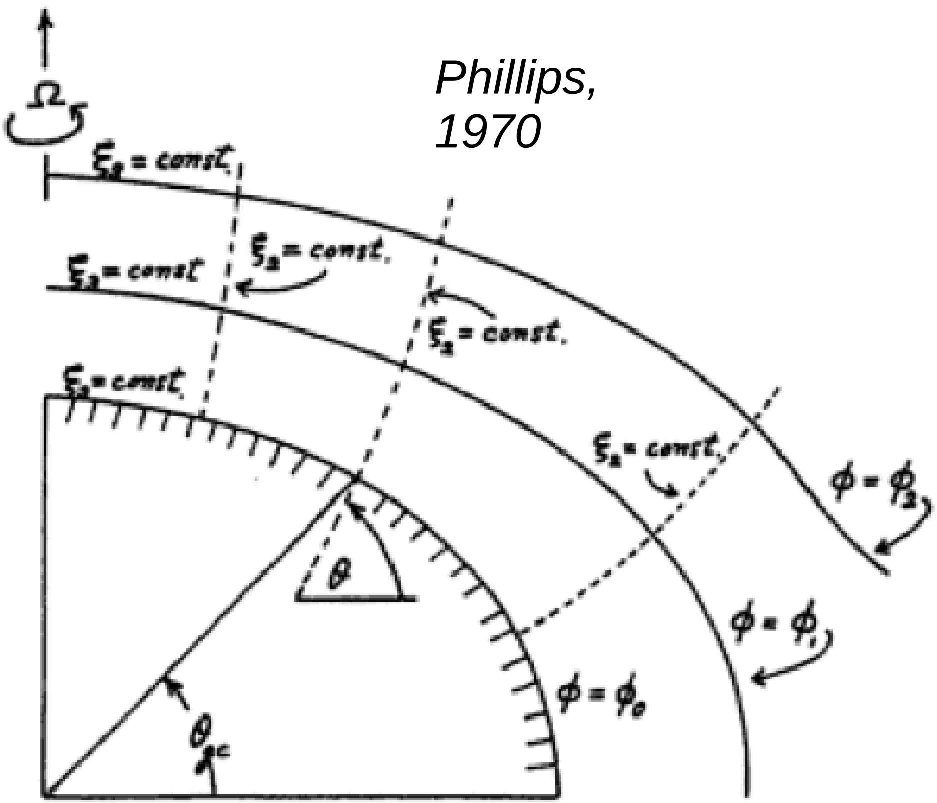
- where by definition

$$u^i = \frac{D\xi^i}{Dt}$$

are the *contravariant* components of velocity

Transport in curvilinear coordinates

Phillips, 1970



Surfaces of constant ξ_2 and constant ξ_3

$$
\begin{aligned}
J(\xi^i)\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 &= \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\mathrm{d}m = \mu\mathrm{d}\xi^1\mathrm{d}\xi^2\mathrm{d}\xi^3 &= \rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
\mu &= \rho J
\end{aligned}
$$

pseudo-density

$$(x,\, y,\, z) \to (\xi^1,\, \xi^2,\, \Phi)$$

$$\dot{x} \qquad u^i = \frac{D\xi^i}{Dt}$$

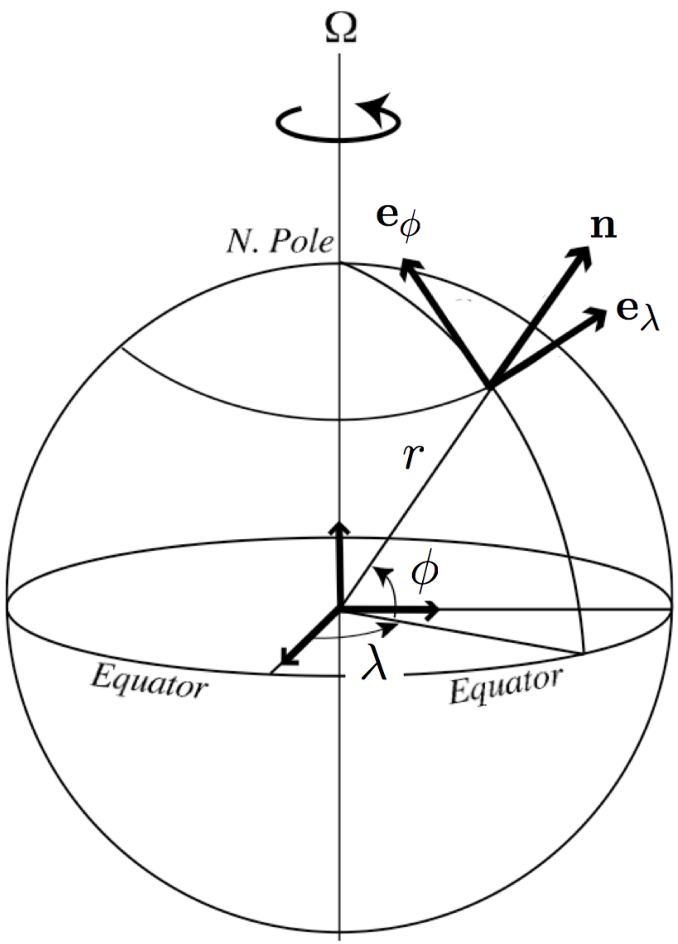
contravariant velocity components

$$\frac{D}{Dt}s(t,\xi^i) = \frac{\partial s}{\partial t} + \frac{D\xi^i}{Dt}\frac{\partial s}{\partial \xi^i} = \left(\frac{\partial}{\partial t} + u^i\partial_i\right)s$$

$$
\begin{aligned}
\frac{\partial \mu}{\partial t} + \partial_i\left(\mu u^i\right) &= 0 \\
\frac{\partial s}{\partial t} + u^i\partial_i s &= \frac{q}{T} \\
\frac{\partial}{\partial t}\left(\mu s\right) + \partial_i\left(\mu s u^i\right) &= \frac{Q}{T}
\end{aligned}
$$

mass and entropy budget
conservative form

Physical vs contravariant components

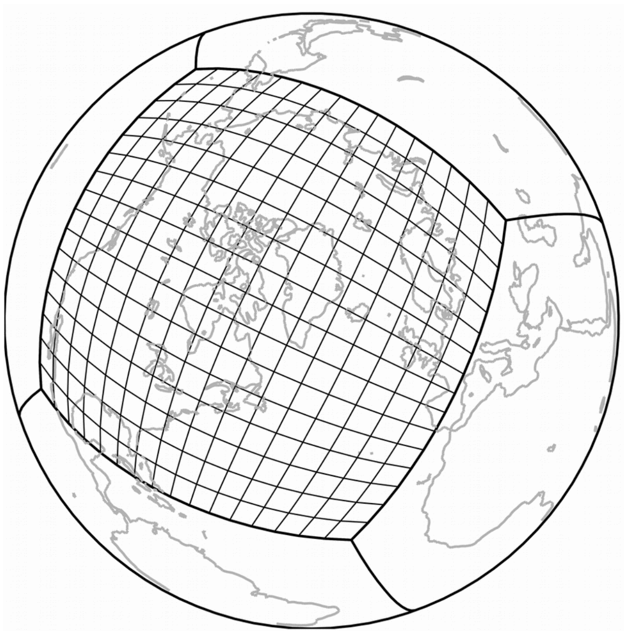


$$
\begin{aligned}
\frac{\partial s}{\partial t} + \frac{u}{r\cos\phi}\partial_\lambda s + \frac{v}{r}\partial_\phi s + w\partial_r s &= \frac{q}{T} \\
\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\phi}\partial_\lambda(\rho u) + \frac{1}{r}\partial_\phi(\rho v) + \frac{1}{r^2}\partial_r(\rho r^2 w) &= 0
\end{aligned}
$$

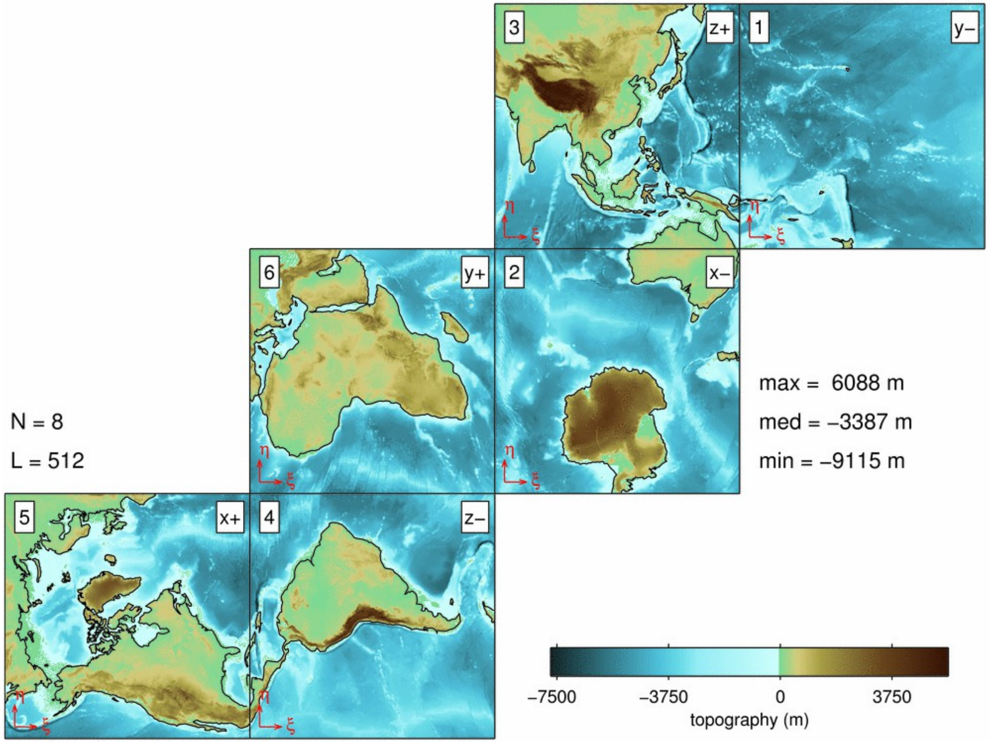
$$
\begin{aligned}
\frac{\partial s}{\partial t} + \dot\lambda\partial_\lambda s + \dot\phi\partial_\phi s + \dot r\partial_r s &= \frac{q}{T} \\
\rho r^2\cos\phi &= \mu \\
\frac{\partial \mu}{\partial t} + \partial_\lambda\left(\mu\dot\lambda\right) + \partial_\phi\left(\mu\dot\phi\right) + \partial_r(\mu\dot r) &= 0
\end{aligned}
$$

- Contravariant formulation independent from the coordinate system
- No information about the geometry needed
- Naturally in conservative form (flux-form)

Physical vs contravariant components



Sadourny, 1972



- Contravariant formulation independent from the coordinate system
- No information about the geometry needed
- Naturally in conservative form (flux-form)

Important consequences of the transport equations

$$\frac{Dq}{Dt} = 0 \quad \Rightarrow \quad \frac{D}{Dt}F(q) = 0$$

$$q = cst \,\mathrm{at}\, t = 0 \quad \Rightarrow \quad q = cst \,\mathrm{at}\, t > 0$$

$$\frac{Dq_1}{Dt} = 0 \ldots \frac{Dq_n}{Dt} = 0 \quad \Rightarrow \quad \frac{D}{Dt}F(q_1, \ldots, q_n) = 0$$

$$F(q_1, \ldots, q_n) = 0 \,\mathrm{at}\, t = 0 \quad \Rightarrow \quad F(q_1, \ldots, q_n) = 0 \,\mathrm{at}\, t > 0$$

Important consequences of the transport equations

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\rho\dot{\mathbf{x}} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\int_{\mathcal{D}} \rho \mathrm{d}^3x + \int_{\partial\mathcal{D}} \rho\dot{\mathbf{x}}\cdot\mathbf{n}\mathrm{d}S = 0$$

$$\frac{Dq}{Dt} = 0 \quad \Rightarrow \quad \frac{\partial\rho q}{\partial t} + \mathrm{div}\rho q\dot{\mathbf{x}} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\int_{\mathcal{D}} \rho q \mathrm{d}^3x + \int_{\partial\mathcal{D}} \rho q\dot{\mathbf{x}}\cdot\mathbf{n}\mathrm{d}S = 0$$

$$\frac{Dq_1}{Dt} = 0 \ldots \frac{Dq_n}{Dt} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\int_{\mathcal{D}} \rho F(q_1,\ldots,q_n)\mathrm{d}^3x + \int_{\partial\mathcal{D}} \rho F(q_1,\ldots,q_n)\dot{\mathbf{x}}\cdot\mathbf{n}\mathrm{d}S = 0$$

Technical reminder :

$$\frac{\partial}{\partial t}\int_{\mathcal{D}} \mathrm{div}\mathbf{F}\mathrm{d}^3x = \int_{\partial\mathcal{D}} \mathbf{F}\cdot\mathbf{n}\mathrm{d}S$$

Geometric approximations :

The shape of the Earth and its approximate representation

THEORIE
DE LA
FIGURE
DE LA TERRE,

Tirée des Principes de l'Hydrostatique.

Par M. CLAIRAUT, de l'Académie Royale des Sciences, & de la Société Royale de Londres.

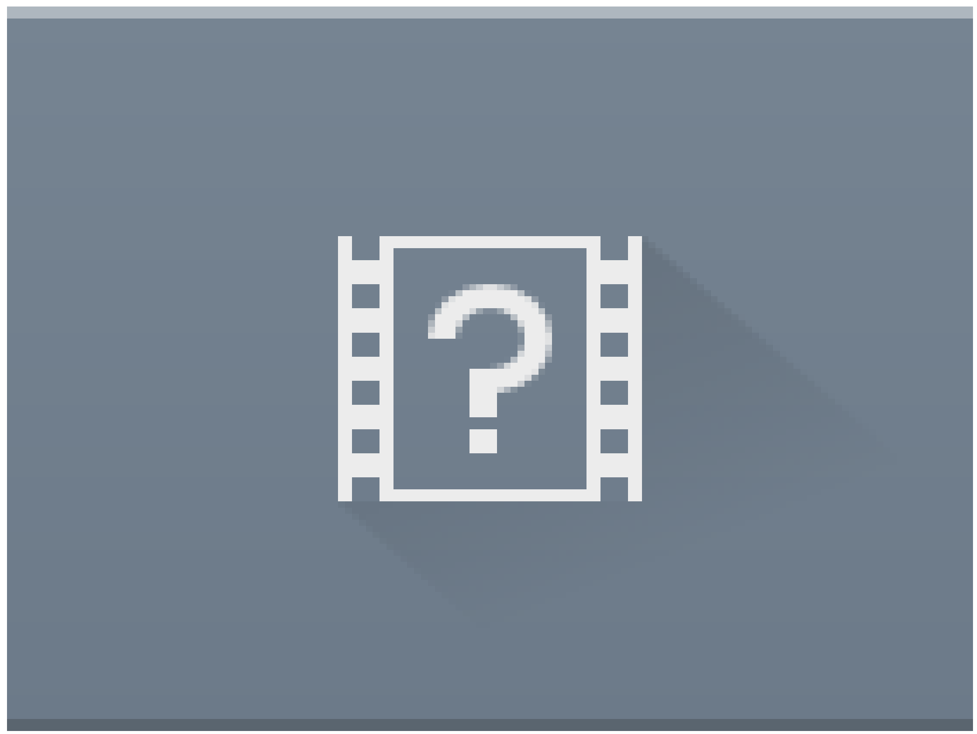
J. L. De Praytorrens.

A PARIS,
Chez DURAND, Libraire, ruë Saint-Jacques, à Saint Landry & au Griffon.

MDCCXLIII.

AVEC APPROBATION ET PRIVILEGE DU ROY.

Motion in an inertial Cartesian frame



$$\frac{D\dot{\mathbf{x}}}{Dt} + \frac{1}{\rho}\nabla p + \nabla V = 0$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\rho\dot{\mathbf{x}} = 0 \qquad \frac{Ds}{Dt} = \frac{q}{T}$$

Motion in a rotating Cartesian frame

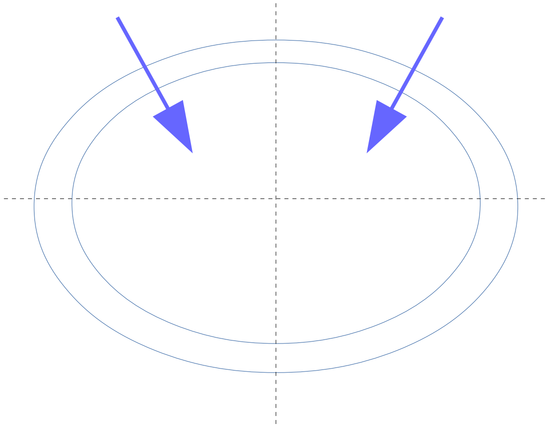
- Newton's fundamental principle of dynamics
- Forces : pressure and gravity
- Pseudo-forces : Coriolis and centrifugal

Planetary velocity

$$\mathbf{R} = \mathbf{\Omega} \times \mathbf{x}$$

Geopotential

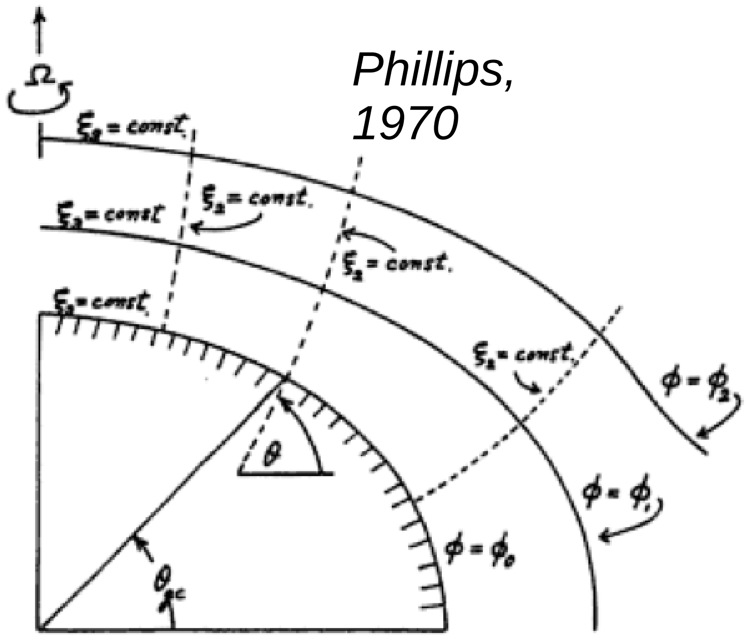
$$\begin{aligned} \Phi &= V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R} \\ \mathbf{g} &= -\nabla\Phi \end{aligned}$$



$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R}\times\dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\ \rho\dot{\mathbf{x}} = 0 \qquad \frac{Ds}{Dt} = \frac{q}{T}$$

Dynamics in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

Phillips, 1970

$$\left(\xi^1, \xi^2, \xi^3\right) \mapsto \mathbf{x}\left(\xi^1, \xi^2, \xi^3\right)$$

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$

$$\mathbf{u}\cdot\mathbf{u} = ?$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

?

$$\begin{aligned} x =& r\cos\phi\cos\lambda \\ y =& r\cos\phi\sin\lambda \\ z =& r\sin\phi \end{aligned}$$

$$\left(\xi^1,\xi^2,\xi^3\right) \mapsto \mathbf{x}\left(\xi^1,\xi^2,\xi^3\right)$$

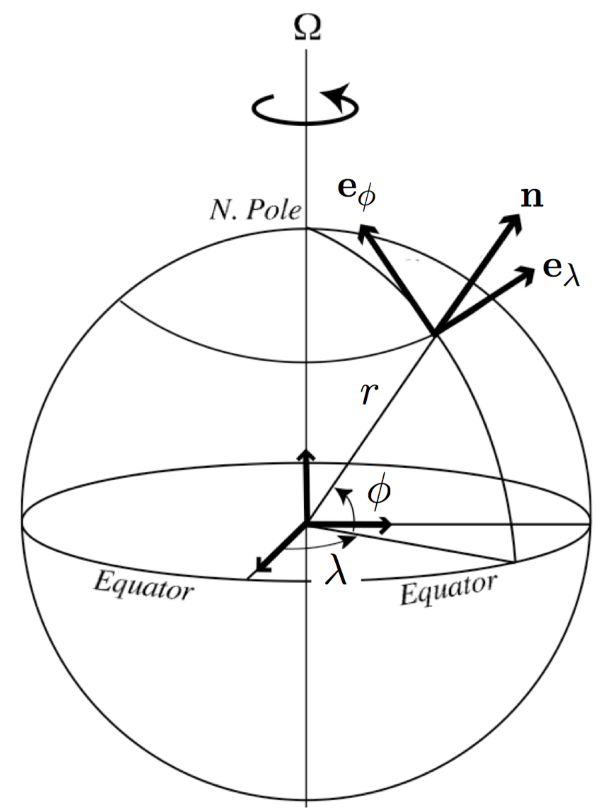
$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$

$$\mathbf{u}\cdot\mathbf{u} = ?$$

$$\mathbf{e}_\lambda \equiv \frac{\partial \mathbf{x}}{\partial \lambda} = (-r\cos\phi\sin\lambda, r\cos\phi\cos\lambda, 0)$$

$$\mathbf{e}_\phi \equiv \frac{\partial \mathbf{x}}{\partial \phi} = (-r\sin\phi\cos\lambda, -r\sin\phi\sin\lambda, r\cos\phi)$$

$$\mathbf{e}_r \equiv \frac{\partial \mathbf{x}}{\partial r} = (\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi)$$



$$\dot{\mathbf{x}} = \dot\lambda\mathbf{e}_\lambda + \dot\phi\mathbf{e}_\phi + \dot r\mathbf{e}_r = u^i\mathbf{e}_i$$

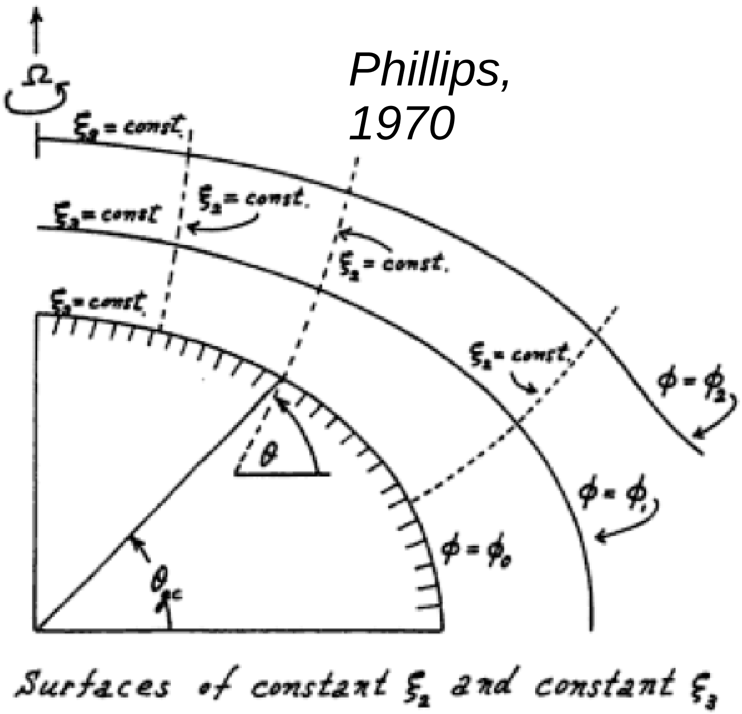
$$\begin{aligned} \dot{\mathbf{x}}\cdot\dot{\mathbf{x}} &= (\dot\lambda\mathbf{e}_\lambda + \dot\phi\mathbf{e}_\phi + \dot r\mathbf{e}_r)\cdot(\dot\lambda\mathbf{e}_\lambda + \dot\phi\mathbf{e}_\phi + \dot r\mathbf{e}_r) \\ &= \dot\lambda^2\mathbf{e}_\lambda\cdot\mathbf{e}_\lambda + \dot\phi^2\mathbf{e}_\phi\cdot\mathbf{e}_\phi + \dot r^2\mathbf{e}_r\cdot\mathbf{e}_r \\ &= G_{ij}u^iu^j = u^iu_i \end{aligned}$$

$$G_{\lambda\lambda} = r^2\cos^2\phi \qquad u_\lambda = r^2\cos^2\phi\,\dot\lambda \qquad R_\lambda = \Omega r^2\cos^2\phi$$

$$G_{\phi\phi} = r^2 \qquad u_\phi = r^2\dot\phi \qquad u_\phi = 0$$

$$G_{rr} = 1 \qquad u_r = \dot r \qquad u_r = 0$$

Dynamics in curvilinear coordinates



$$\left(\xi^1, \xi^2, \xi^3\right) \mapsto \mathbf{x}\left(\xi^1, \xi^2, \xi^3\right)$$

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt}$$

$$\mathbf{u}\cdot\mathbf{u} = ?$$

$$\begin{aligned} \mathbf{e}_i &= \partial_i \mathbf{x} \\ G_{ij}(\xi^k) &= \mathbf{e}_i \cdot \mathbf{e}_j \\ J &= (\mathbf{e}_1 \times \mathbf{e}_2)\cdot \mathbf{e}_3 \\ &= \sqrt{\det G_{ij}} \end{aligned}$$

metric

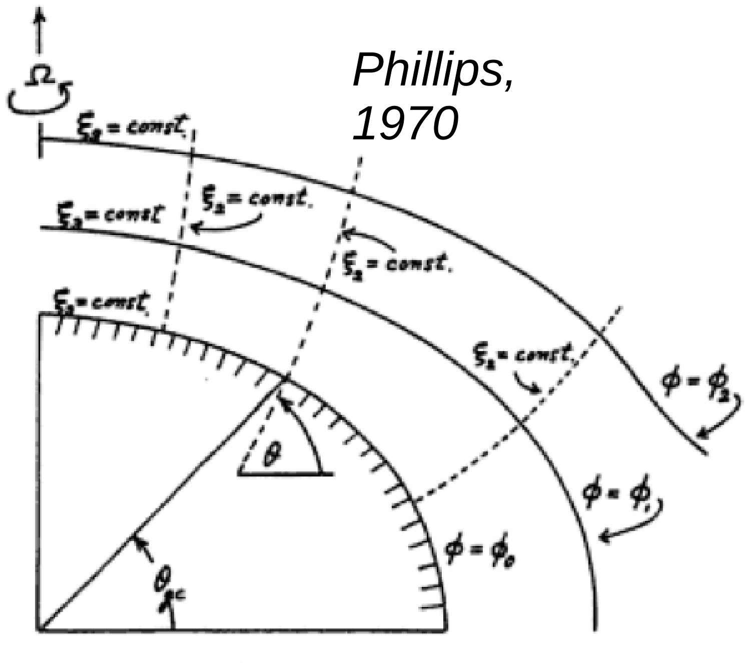
$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla \Phi = 0$$

?

$$\begin{aligned} \mathbf{u} = \frac{D\mathbf{x}}{Dt} &= u^i \mathbf{e}_i \\ \mathbf{u}\cdot\mathbf{e}_i &= G_{ij}u^j = u_i \\ \mathbf{u}\cdot\mathbf{u} &= G_{ij}u^i u^j = u^i u_j \\ \mathbf{R}\cdot\mathbf{u} &= R_i u^i \end{aligned}$$

covariant components

Dynamics in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

Phillips, 1970

- *covariant* : same form in all coordinate systems
- derives from a *variational principle* : Hamilton's principle of least action
- *dynamically consistent* *(White & Bromley, 1995)* : conserves energy, angular momentum, potential vorticity for any choice of zonally-symmetric metric, planetary velocity, Jacobian

metric, planetary velocity, Jacobian can be approximated without jeopardizing *dynamical consistency*

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

various geometric approximations, each characterized by a certain choice of metric and planetary velocity

$$\boxed{\begin{aligned} G_{ij}\frac{Du^j}{Dt} + \frac{1}{2}\left(\partial_j G_{ik} + \partial_k G_{ij} - \partial_i G_{jk}\right) u^j u^k & \\ + \left[\partial_j R_i - \partial_i R_j\right] u^j + \frac{J}{\mu}\partial_i p + \partial_i \Phi &= 0 \end{aligned}}$$

(Tort & Dubos, 2014)

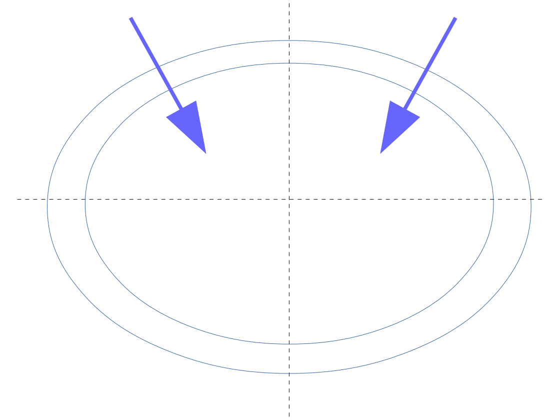
Spherical geoid approximation

- Ellipticity of geoids ~ centrifugal / gravitational ~ 1/300
- Spherical geoid approximation : pretend that the metric in geopotential coordinates is actually spherical !

$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j = r^2\cos^2\phi\,\mathrm{d}\lambda^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}r^2 \qquad R_j\mathrm{d}\xi^j = \Omega r^2\sin^2\phi\,\mathrm{d}\lambda$$

Geopotential

$$\begin{aligned}\Phi &= V - \frac{1}{2}\mathbf{R}\cdot\mathbf{R}\\ \mathbf{g} &= -\nabla\Phi\end{aligned}$$



East-west wind velocity

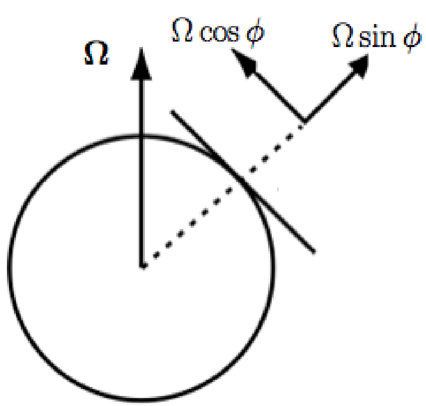
latitude

Pressure

$$D_t u - \left(2\Omega + \frac{u}{r\cos\phi}\right) v \sin\phi + 2\Omega\cos\phi\, w + \frac{uw}{r} + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$D_t v + \left(2\Omega + \frac{u}{r\cos\phi}\right) u \sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$D_t w + 2\Omega\cos\phi\, u - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$



South-North wind velocity

	Earth	Titan
ratio	$\sim 1\%$	$\sim 25\%$
g variability	$\sim 2.8\%$	$\sim 53.4\%$

- atmospheric shallowness (radius a=6400 km >> r-a~50 km) suggests to let r=a, g(r)=g(a) and neglect blue terms : shallow-fluid approximation
- Small vertical velocities suggest to neglect red terms : traditional approximation

$$D_t u - \left(2\Omega + \frac{u}{r\cos\phi}\right) v \sin\phi + 2\Omega\cos\phi w + \frac{uw}{r} + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$D_t v + \left(2\Omega + \frac{u}{r\cos\phi}\right) u \sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$D_t w + 2\Omega\cos\phi u - \frac{u^2+v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$

	Earth	Titan
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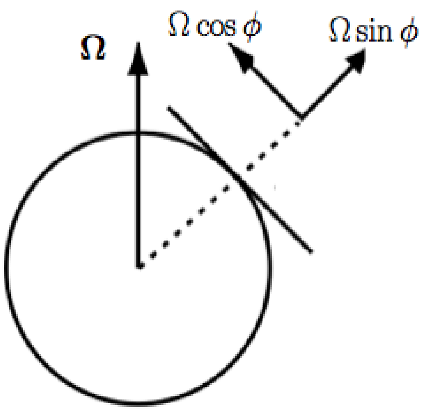
TAB 1. *Data from NASA website*



- atmospheric shallowness (radius a=6400 km >> r-a~50 km) suggests to let r=a, g(r)=g(a) and neglect blue terms : shallow-fluid approximation
- Small vertical velocities suggest to neglect red terms : traditional approximation
- Phillips (1966) : both approximations must be made together, otherwise angular momentum conservation is lost

Shallow-fluid : replace r by a in the metric

Traditional : replace r by a in the planetary velocity (Tort & Dubos, 2014)

$$D_t u - \left(2\Omega + \frac{u}{r\cos\phi}\right) v \sin\phi + 2\Omega\cos\phi w + \frac{uw}{r} + \frac{1}{\rho r\cos\phi}\partial_\lambda p = 0$$

$$D_t v + \left(2\Omega + \frac{u}{r\cos\phi}\right) u \sin\phi + \frac{vw}{r} + \frac{1}{\rho r}\partial_\phi p = 0$$

$$D_t w + 2\Omega\cos\phi u - \frac{u^2 + v^2}{r} + g + \frac{1}{\rho}\partial_r p = 0$$

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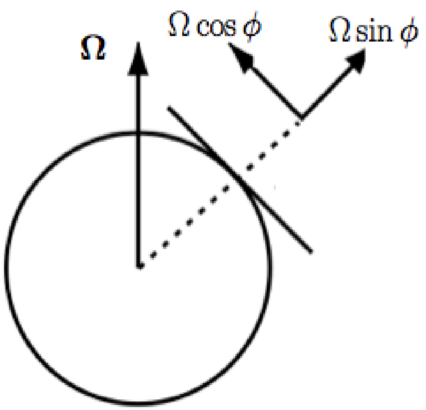


Shallow-fluid and traditional approximations

$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j \equiv r^2\cos^2\phi\mathrm{d}\lambda^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}r^2$$

$$R_j\mathrm{d}\xi^j \equiv \Omega r^2\sin^2\phi\mathrm{d}\lambda$$

	Earth	Titan
ratio	$\sim 1\%$	$\sim 25\%$
g variability	$\sim 2.8\%$	$\sim 53.4\%$

Atmospheric/oceanic shallowness (radius a=6400 km >> r-a~50 km) suggests to let r=a

Shallow-fluid : replace r by a in the metric

Traditional : replace r by a in the planetary velocity (Tort & Dubos, 2014)

$$G_{ij}\mathrm{d}\xi^i\mathrm{d}\xi^j \equiv a^2\cos^2\phi\mathrm{d}\lambda^2 + a^2\mathrm{d}\phi^2 + \mathrm{d}z^2$$

$$r = a + z$$

$$\Phi = gz$$

$$R_j\mathrm{d}\xi^j \equiv \Omega a^2\sin^2\phi\mathrm{d}\lambda$$

$$\Rightarrow \quad \nabla \times R = f\mathbf{e}_z$$

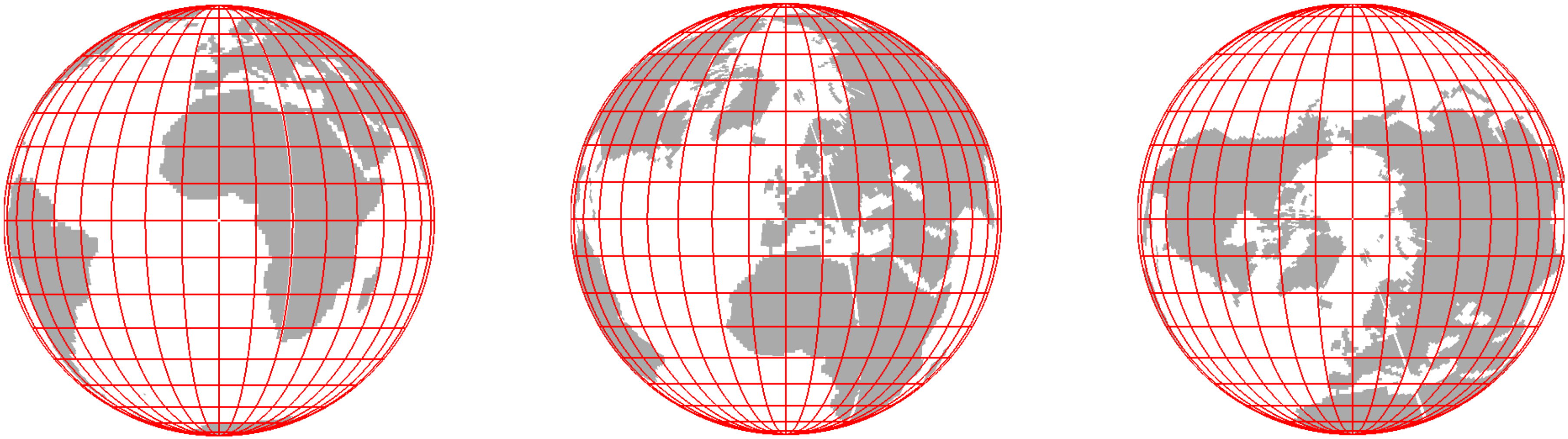
f : Coriolis parameter

$$\frac{D\dot{\mathbf{x}}}{Dt} + f\,\mathbf{e}_z \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

Traditional approximation : $\mathrm{curl}\,\mathbf{R} = f\,\mathbf{e}_z$

=> only the Coriolis parameter f matters for the dynamics

Why would the pole of the coordinate system be at the geographical pole ?
Let us put it at some arbitrary latitude … (Verkley, 1990)



$$f(\lambda,\phi) = 2\Omega\sin\phi$$

$$f(\lambda,\phi) = 2\Omega\cos\lambda\cos\phi$$

$$f(\lambda,\phi) = 2\Omega\,(\sin\phi\cos\phi_0 + \cos\lambda\cos\phi\sin\phi_0)$$

Tangent-plane approximations

Cartesian models with a simple expression of f

$$\frac{D\dot{\mathbf{x}}}{Dt} + f\,\mathbf{e}_z \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$

Equatorial beta-plane

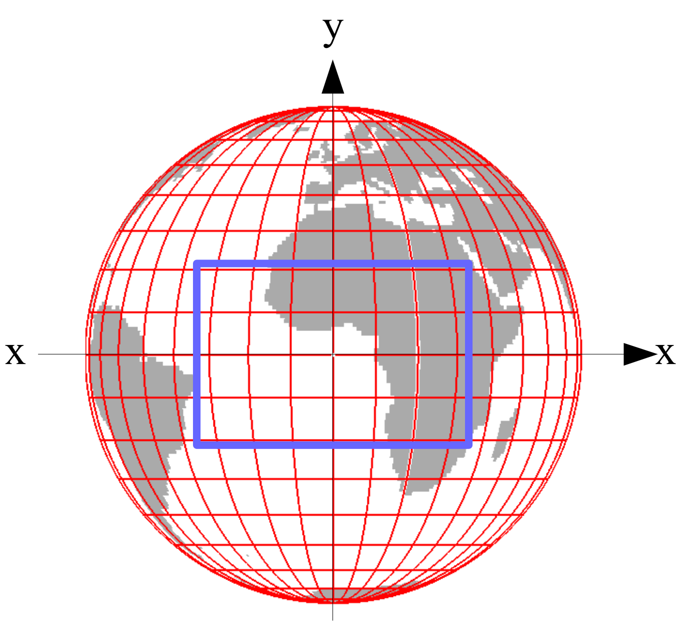
$$f = \beta y$$

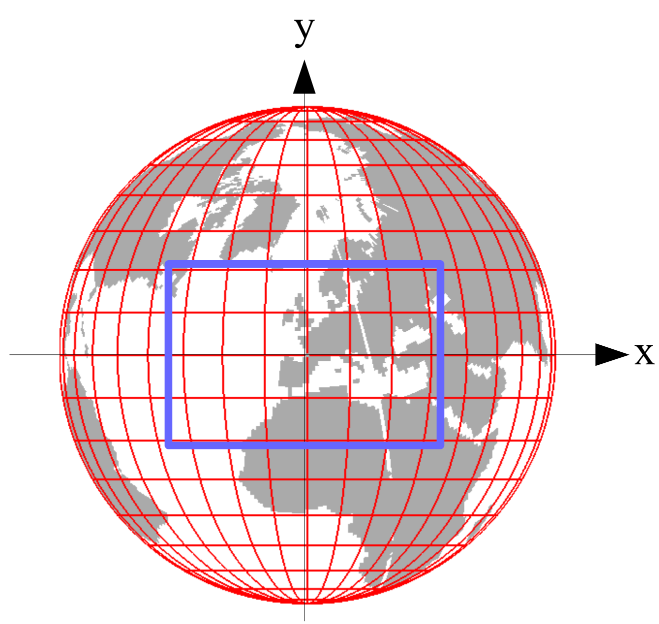
Beta-plane
f-plane

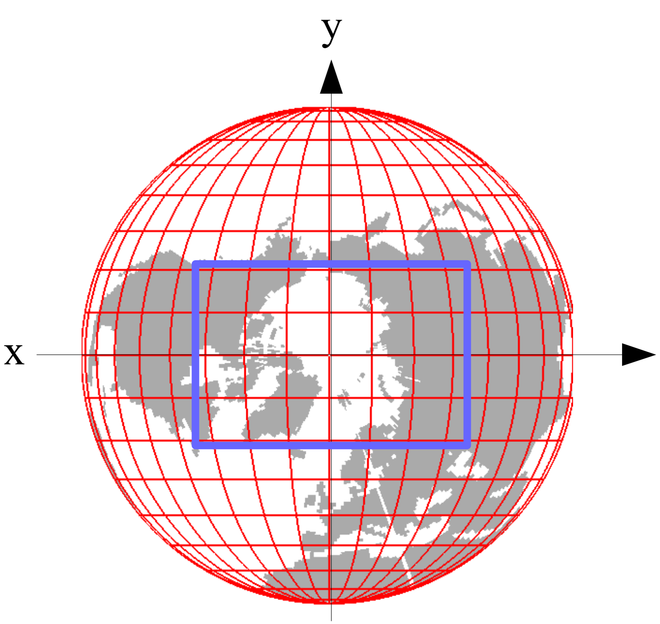
$$f = f_0 + \beta y$$

Delta-plane

$$f = 2\Omega\left(1 - \frac{x^2 + y^2}{2a^2}\right)$$







$$f(\lambda,\phi) = 2\Omega\sin\phi$$

$$\phi = \frac{y}{a} \ll 1, \quad \lambda = \frac{x}{a} \ll 1$$

$$f(\lambda,\phi) = 2\Omega\cos\lambda\cos\phi$$

$$f(\lambda,\phi) = 2\Omega\left(\sin\phi\cos\phi_0 + \cos\lambda\cos\phi\sin\phi_0\right)$$

Wrap-up

Inertial Cartesian ↔ Rotating Cartesian ↔ Rotating Curvilinear → Rotating Curvilinear

Gravity

Planetary velocity
Geopotential
Coriolis

Contravariant
Covariant

Exact geometry

↓

spherical geoid

↓

shallow-fluid traditional

↓

beta-plane

↓

f-plane

Fully compressible fluid

↓ *Boussinesq/anelastic* ↓ *Hydrostatic*

↓ *Hydrostatic Boussinesq*

- Geometry = metric tensor + planetary velocity
- Affects the equations of dynamics (momentum)
- But **transport equations** are the same in **any coordinate system**
- Flux-form <=> advective form

References

Phillips (1966)	J. Atmos. Sci **23**: 626-628
Verkley (1990)	J. Atmos. Sci **47**(20): 2453-2460
White & Wood (2012)	Q. J. R. Meteorol. Soc. **138**: 980 – 988,
Tort & Dubos (2014)	J. Atmos. Sci **71**(7): 2452-2466
Bénard (2015)	Q. J. R. Meteorol. Soc. **141** (686): 195-206