Dynamical approximations

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Energy
Momentum
Potential vorticity

Inertial Cartesian Eulerian	⟺	Rotating Cartesian Eulerian	⟺	Rotating Curvilinear Eulerian	⟹	Rotating Curvilinear Eulerian	⟺	Rotating Curvilinear non-Eulerian
Thermodynamics *Gravity*		*Planetary velocity* *Geopotential* *Coriolis*		*Kinematics* *Covariant* *Contravariant*		*Spherical geoid* *Shallow-fluid* *Traditional* *Boussinesq/anelastic* *Hydrostatic*		*Terrain-following* *Lagrangian* *Isentropic/isopycnal* *mass-/pressure-based*

Dynamical effects of thermodynamics

The atmosphere and ocean are *stratified* flows

Consider the simple question of the stability of a resting atmosphere/ocean

- Naive criterion : density decreases with altitude
- Time scale of buoyancy oscillations : Brunt-Vaisala frequency N

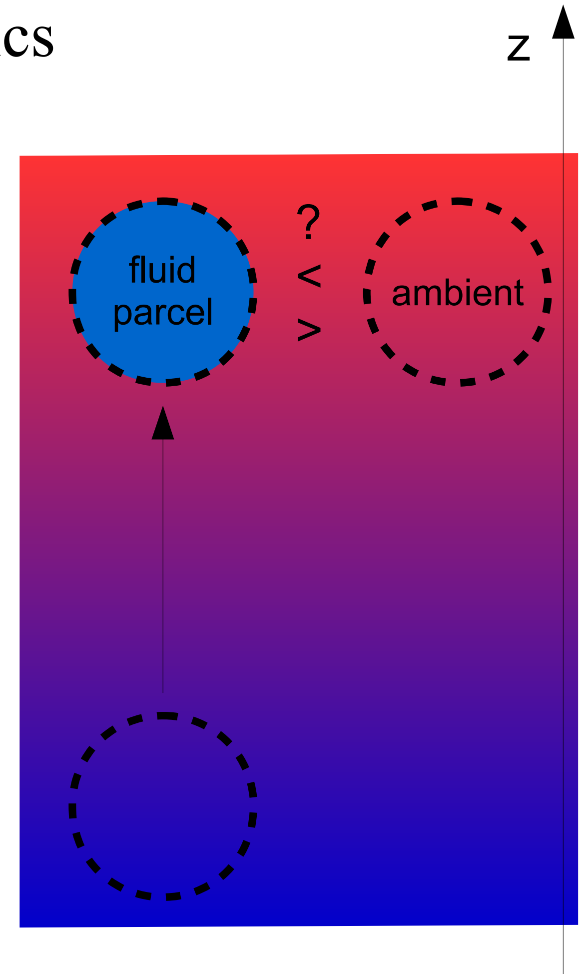
$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} > 0$$

- In fact : fluid parcels adjust rapidly and adiabatically their density to ambient pressure

$$\Rightarrow \qquad N^2 = -\frac{g}{\rho}\left(\frac{\mathrm{d}\rho}{\mathrm{d}z} - \frac{\mathrm{d}p}{\mathrm{d}z}\frac{\partial\rho}{\partial p}\right)$$

- atmospheric scientists use potential temperature

$$N^2 = -\frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z} > 0$$



density

$$\rho = \rho(s, p, q)$$

Mass fraction (salinity)

Specific entropy

pressure

temperature

$$\theta = T(s, p_0)$$

Important to have a *simple*, *correct* and *general* approach to thermodynamics

Equation of state of a compressible fluid

Commonly encountered approach (ideal perfect gas) :

- equation of state relates pressure, density and temperature
- specific heat defines internal energy
- potential temperature used to characterize adiabatic transforms
- complemented by a bunch of other relationships

$$
\begin{aligned}
p &= \rho R T \\
e &= c_v T \\
\theta &= T \left(\frac{p}{p_0} \right)^{-R/c_p} \\
c_p - c_v &= R
\end{aligned}
$$

Pro :

- simple
- avoids reviving bad memories about entropy, second principle, Maxwell relationships, ...

Con :

- « accidental » relationships
- cumbersome for non-ideal gases (variable cp)
- cumbersome for mixtures (moist air, salty water)
- overall *energetic consistency*

$$
\begin{aligned}
\pi &= c_p \left(\frac{p}{p_0} \right)^{R/c_p} \\
e + \frac{p}{\rho} &= c_p T = \theta \pi \\
\frac{1}{\rho} \mathrm{d}p &= \theta \mathrm{d}\pi
\end{aligned}
$$

Thermodynamics of a compressible fluid

Systematic approach (Ooyama, 1990 ; Bannon, 2003 ; Feistel, 2008)

- state variables :
  - specific volume/pressure
  - specific entropy / temperature
  - mixing ratio / chemical potential (mixtures)
- All relations follow from the expression of a single thermodynamic potential

$$\mathrm{d}e = -p\mathrm{d}\alpha + T\mathrm{d}s + \mu\mathrm{d}q$$

$$\Rightarrow \quad e(\alpha, s, q)$$

$$\alpha \leftrightarrow p \qquad s \leftrightarrow T \qquad q \leftrightarrow \mu$$

Specific enthalpy

$$h(p, s, q) \equiv e + \alpha p$$

Specific Gibbs function

$$g(p, T, q) \equiv h - Ts$$

Pro :
- always energetically consistent
- general : variable cp, mixtures

Con :
- none
- unless you *really* hate thermodynamics

$$\mathrm{d}h = \alpha\mathrm{d}p + T\mathrm{d}s + \mu\mathrm{d}q$$

$$\mathrm{d}g = \alpha\mathrm{d}p - s\mathrm{d}T + \mu\mathrm{d}q$$

Ideal perfect gas

$$h(p,s) = h_0 - C_pT_0 + C_pT_0\left(\frac{p}{p_0}\right)^{R/C_p}\exp\frac{s-s_0}{C_p}$$

$$T(p,s) = \frac{\partial h}{\partial s} = T_0\left(\frac{p}{p_0}\right)^{R/C_p}\exp\frac{s-s_0}{C_p}$$

$$\mathrm{d}h = C_p\mathrm{d}T$$

$$\theta = T(p_0,s) = T_0\exp\frac{s-s_0}{C_p}$$

$$\theta = T\left(\frac{p}{p_0}\right)^{R/C_p}$$

$$\alpha(p,s) = \frac{\partial h}{\partial p} = \frac{R}{C_p}\frac{h-(h_0-C_pT_0)}{p}$$

$$p = \rho RT$$

Incompressible fluid

$$h(p, s, q) = h_0(s, q) + \alpha(s, q)(p - p_0)$$

$$\alpha = \frac{\partial h}{\partial p} = \alpha(s, q)$$

Density varies only due to heating / salinity

$$e = h - \alpha p = h_0 - \alpha(s, q) p_0$$

Internal energy is degenerate
=> use enthalpy to describe nearly incompressible fluids

$$T = \frac{\partial h}{\partial s} = \frac{\partial h_0}{\partial s} + \frac{\partial \alpha_0}{\partial s}(p - p_0)$$

$$\frac{\partial T}{\partial p} = \frac{\partial \alpha}{\partial s}$$

$$T(p_0, s) = \frac{\partial h_0}{\partial s}$$

Potential temperature

$$h(p_0, s, q) = h_0(s, q)$$

Potential enthalpy
~ conservative temperature

Lagrangian least action principle for fluid flow

(Eckart, 1960 ; Morrison, 1998)

$$\rho = \rho(p, s)$$

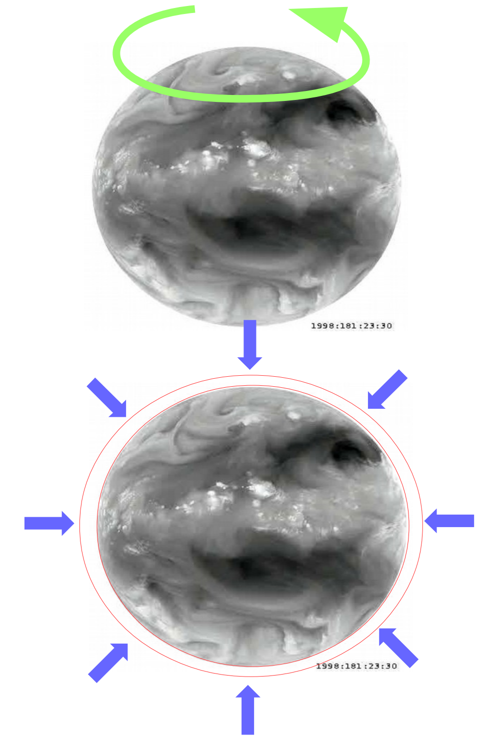
$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

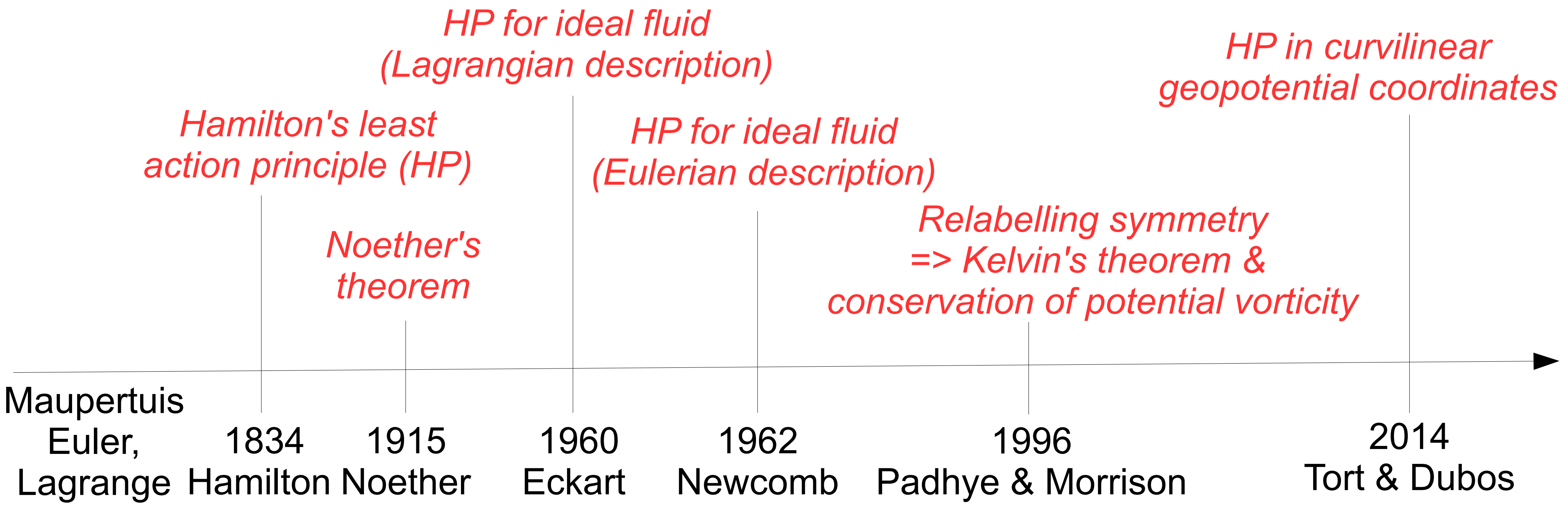
$$\Updownarrow$$

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0$$

$$\frac{\partial L}{\partial p} = 0$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s, p) = \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{2} + \mathbf{R} \cdot \dot{\mathbf{x}} - \Phi - h\,(p, s) + \frac{p}{\rho}$$




HP for ideal fluid
(Lagrangian description)
HP in curvilinear
geopotential coordinates
Hamilton's least
action principle (HP)
HP for ideal fluid
(Eulerian description)
Relabelling symmetry
=> Kelvin's theorem &
conservation of potential vorticity
Noether's
theorem
Maupertuis
Euler,
Lagrange
1834
Hamilton
1915
Noether
1960
Eckart
1962
Newcomb
1996
Padhye & Morrison
2014
Tort & Dubos

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0$$

$$\mathbf{v} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$$

conjugate momentum
= absolute velocity

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) - \nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) + \mathrm{div}\ \left[\rho \dot{\mathbf{x}} \otimes \mathbf{v}\right] = \rho \frac{\partial L}{\partial \mathbf{x}}$$

Flux-form momentum budget

$$E = \dot{\mathbf{x}} \cdot \mathbf{v} - L$$

energy

$$\frac{\partial}{\partial t}(\rho E) + \mathrm{div}\ \left[\left(E - \rho \frac{\partial L}{\partial \rho}\right)\rho \dot{\mathbf{x}}\right] = 0$$

Flux-form energy budget

$$B = \mathbf{v} \cdot \dot{\mathbf{x}} - \rho \frac{\partial L}{\partial \rho} - L$$

Bernoulli function

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\nabla \times \mathbf{v}}{\rho} \times \rho \mathbf{u} + \nabla B - T \nabla s = 0$$

Crocco's theorem = curl-form

$$q = \frac{1}{\rho}(\nabla \times \mathbf{v}) \cdot \nabla s$$

$$\frac{Dq}{Dt} = 0$$

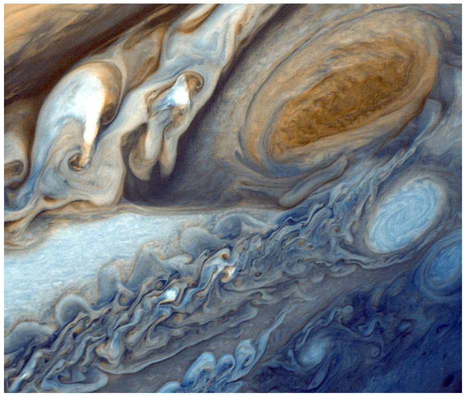
Potential vorticity

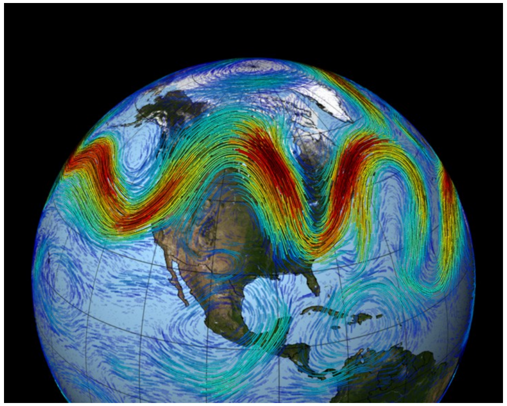
Characteristic scales

- *Velocity* : Sound $c \sim 340 m/s$ Wind $U \sim 30 m/s$
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2} s^{-1}$ Coriolis $f \sim 10^{-4} s^{-1}$
- *Length* : **Scale height $H=c^2/g=10km$** **Rossby radius : $R=c/f \sim 1000$ km**

Mach number : $M=U/c <<1$

Scale separation : $f/N \sim H/R << 1$



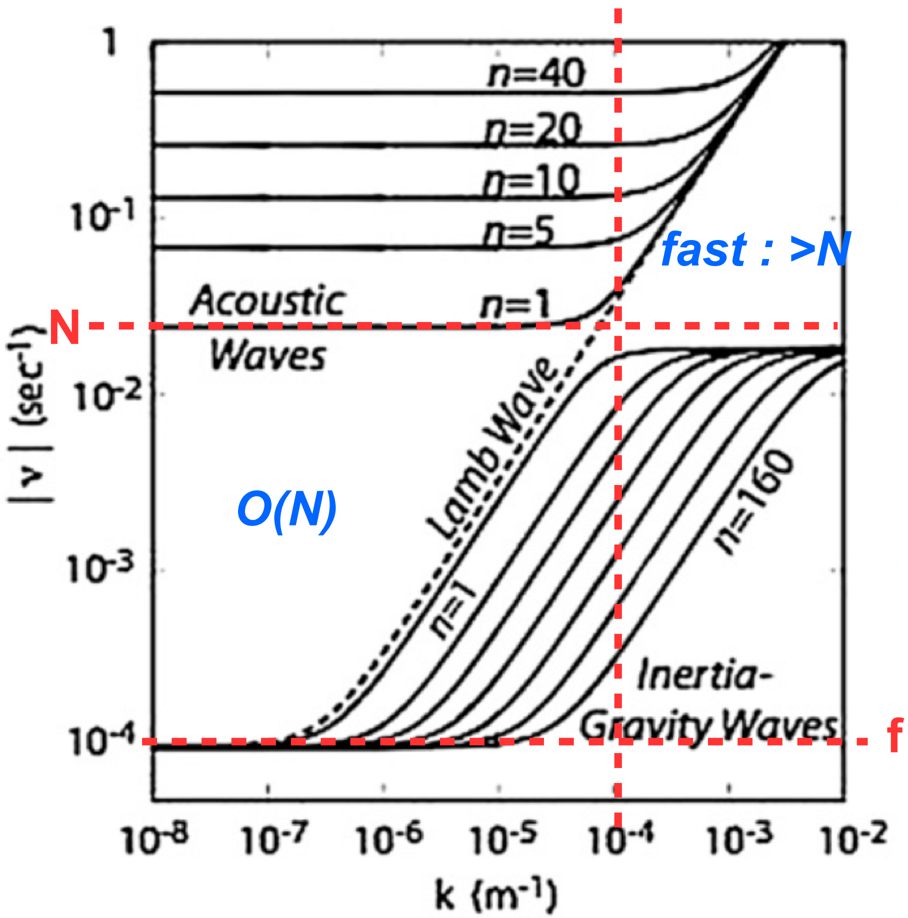


small-scale | scale height | mesoscale | synoptic | planetary

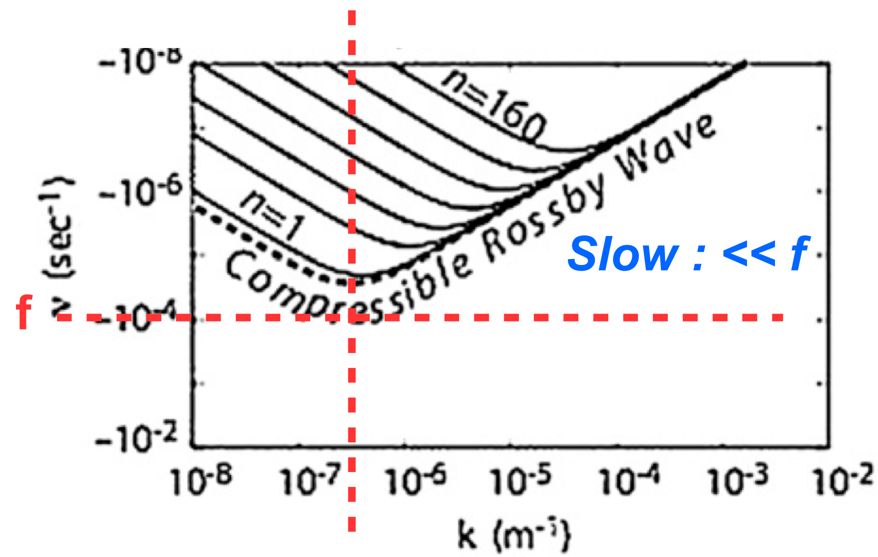
1 km | 10 km | 100 km | 1000 km | 10000 km

Classes of motion in a gravity-dominated, rotating, compressible flow :

Waves in an isothermal atmosphere at rest



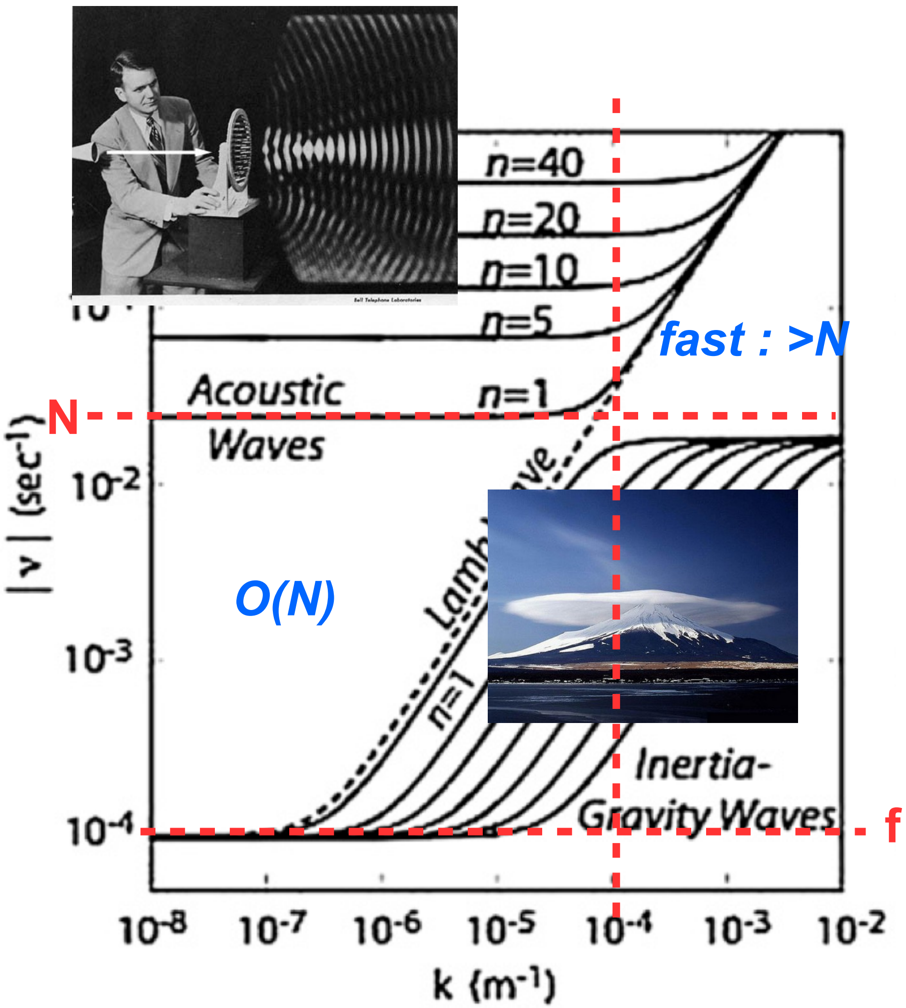
f-plane



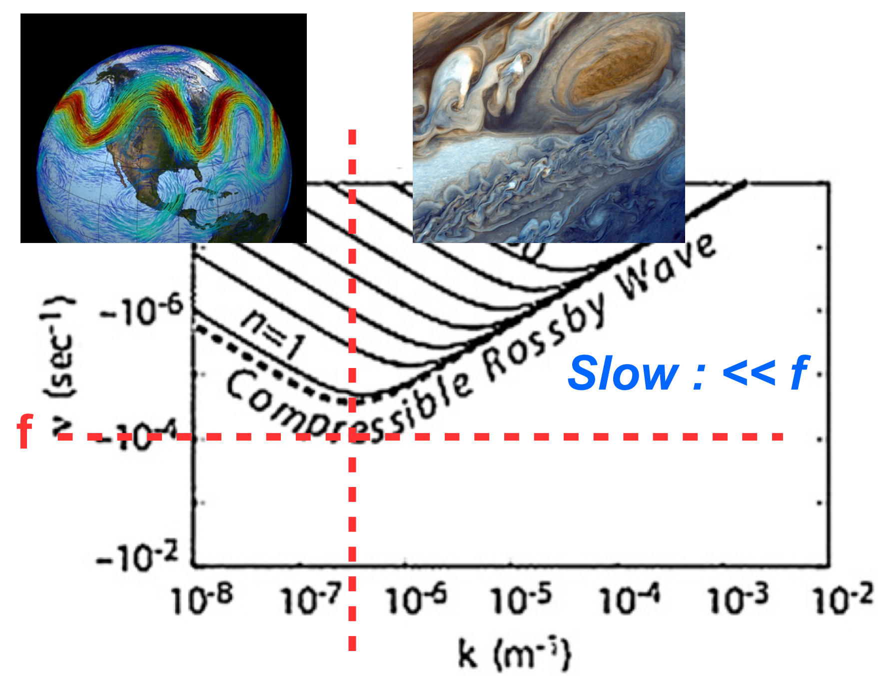
beta-plane

Classes of motion in a gravity-dominated, rotating, compressible flow :

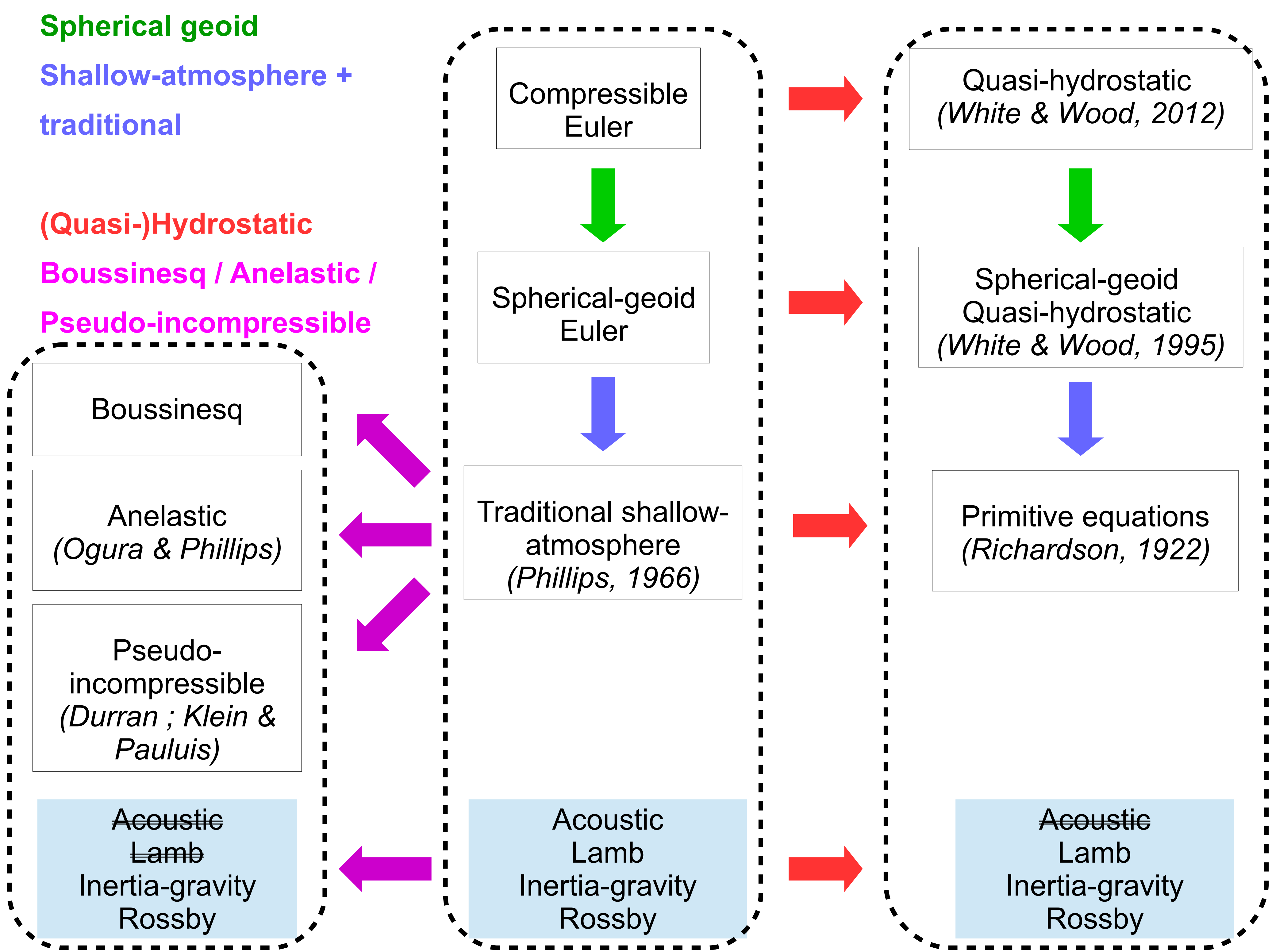
Waves in an isothermal atmosphere at rest



f-plane



beta-plane


Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere (Phillips, 1966)
Acoustic Lamb Inertia-gravity Rossby
Quasi-hydrostatic (White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)
Primitive equations (Richardson, 1922)
~~Acoustic~~ Lamb Inertia-gravity Rossby
Boussinesq
Anelastic (Ogura & Phillips)
Pseudo-incompressible (Durran ; Klein & Pauluis)
~~Acoustic~~ ~~Lamb~~ Inertia-gravity Rossby

	NEMO	ROMS	IFS/ARPEGE	MesoNH	WRF	EndGAME	LMDZ	DYNAMICO
Geometry	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG	SG+TSA	SG+TSA
Dynamics	HB	HB	FCE	A	FCE	FCE	HPE	HPE/(FCE)
Grid	CC	CC	LL	CC	CC	LL	LL	HEX
Disc. Dyn	FD	FV	SP	FD	FV	FD	FD	FD
Transport	FV	FV	SL	FV	FV	FV	FV	FV
Conserv.	M, E/Z	M		M	M	M	M, E/Z	M, E
Time	Split-EX	Split-EX	SI	EX	Split-HEVI	SI	EX	EX/HEVI
Helmholtz			Direct	Direct		Iter		

SG	Spherical-Geoid
TSA	Traditional Shallow-Atmosphere
FCE	Fully Compressible Euler
HPE	Hydrostatic Primitive Eq.
HB	Hydrostatic Boussinesq
A	Anelastic
EX	Explicit
SI	Semi-Implicit
Split	Split
HEVI	Horizontally Explicit, Vertically Implicit
Direct	Direct (spectral)
Iter	Iterative

CC	Cartesian Curvilnear
LL	Latitude-Longitude
HEX	Icosahedral-Hexagonal
FD	Finite Difference
FV	Finite Volume
FE	*Finite Element*
SE	*Spectral Element*
SL	Semi-Lagrangian
SP	Spectral
M	Mass and scalars
E	Energy
Z	Enstrophy

References

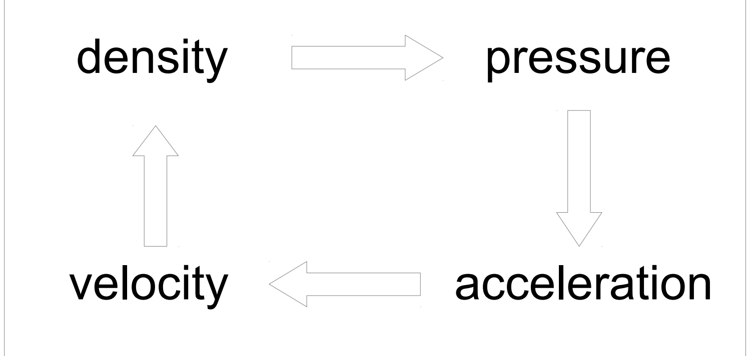
Bannon (2003)	J. Atmos. Sci **60** : 2809-2819
Feistel (2008)	Deep-Sea Res. **55** : 1639-1671
Marquet (2011)	Quart. J. Roy. Met. Soc. **137 :** 768-791

Boussinesq approximations

- (In)compressiblity, acoustic waves, and pressure
- Boussinesq approximation : principle and variants
- Accuracy of wave propagation

(In)compressibility, acoustic waves and pressure

- The equation of state yields pressure given density and specific entropy (and moisture / salinity)



Propagation of acoustic waves

- What happens if the fluid is incompressible ?

$$\frac{1}{\rho} = \alpha(s, r)$$

- Breaks the pressure-density feedback loop
- **Suppresses acoustic waves**
- But how do we determine pressure ??

$$h = h_0(s, r) + \alpha(s, r)(p - p_0)$$

$$L = \dots + p\left(\frac{1}{\rho} - \alpha(s)\right)$$

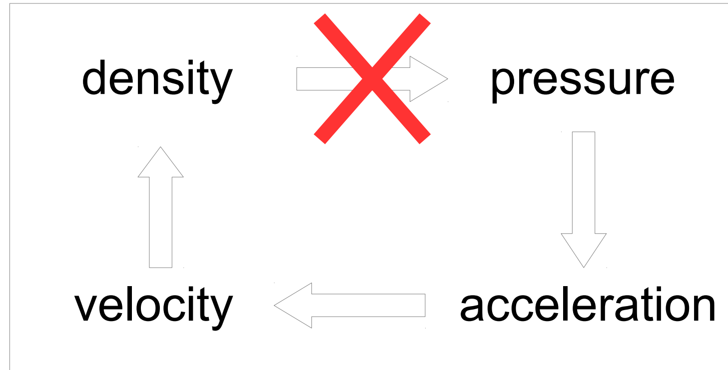
$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0$$

$$\frac{\partial L}{\partial p} = 0$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s, p) = \dots - h(p, s) + \frac{p}{\rho}$$

(In)compressibility, acoustic waves and pressure

- Incompressibilty **contrains** density to depend on entropy, salinity
- The Lagrangian is linear in pressure : **pressure is a Lagrange multiplier** enforcing the constraint ; it is must be determined by additional, **hidden constraints,** to be discovered



Suppression of acoustic waves

Known constraint (incompressibility) :

$$\rho\alpha(s,q) = 1$$

Using kinematics (transport) yields a

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\,\dot{\mathbf{x}} = 0$$

$$\frac{\partial}{\partial t}$$

First hidden constraint :

$$\mathrm{div}\,\dot{\mathbf{x}} = 0$$

Using dynamics yields a

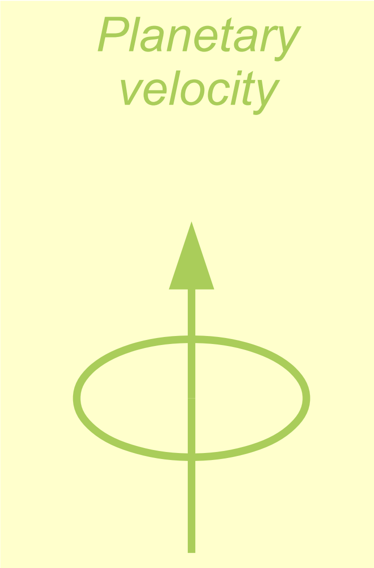
$$\frac{\partial \dot{\mathbf{x}}}{\partial t} + \frac{1}{\rho}\nabla p + \cdots = 0$$

$$\frac{\partial}{\partial t}$$

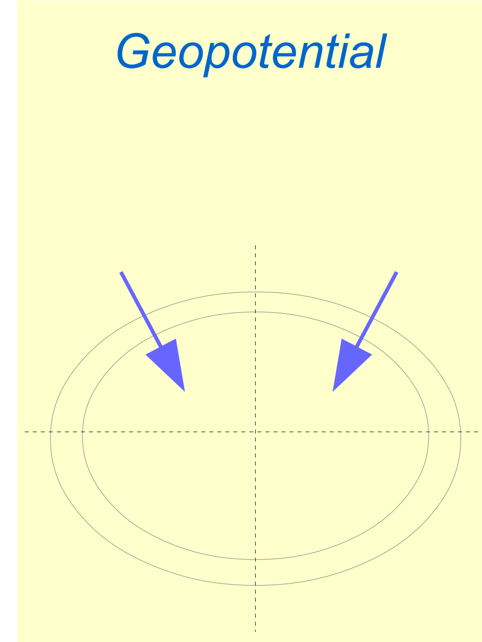
Second hidden constraint :

$$\nabla \cdot \frac{1}{\rho}\nabla p = \ldots$$

This elliptic problem yields p but may be hard/expensive to solve.



$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$



Basic idea of Boussinesq approximations :
pressure remains close to a fixed reference profile

$$p = \overline{p}(\Phi) + p'$$

$$\overline{\rho}(\Phi) \equiv -\frac{\partial \overline{p}}{\partial \Phi}$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = \underbrace{-\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}}_{\text{Buoyancy force}}$$

Basic idea of Boussinesq approximations :
pressure remains close to a fixed reference profile

$$p = \overline{p}(\Phi) + p'$$

$$\overline{\rho}(\Phi) \equiv -\frac{\partial \overline{p}}{\partial \Phi}$$

$$\overline{\rho} = \rho(\overline{p}, \overline{s})$$

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \underbrace{\frac{1}{\rho}}_{\text{Inertia}}\nabla p' = \underbrace{-\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}}_{\text{Buoyancy}}$$

$$\rho(\overline{p} + p', s) = \underbrace{\rho(\overline{p}, \overline{s})}_{\overline{\rho}(\Phi)} + \underbrace{\rho(\overline{p}, s)}_{\rho^*(\Phi, s)} - \rho(\overline{p}, \overline{s}) + \underbrace{\rho(\overline{p} + p', s) - \rho(\overline{p}, s)}_{\simeq \rho' = c^{-2}p'}$$

Reference density varies with altitude / depth

Warmer air rises, colder water sinks

Density fluctuates due to pressure variations caused by flow (dynamic pressure)

Basic idea of Boussinesq approximations :
pressure remains close to a fixed reference profile

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = \underbrace{-\left(\frac{\overline{\rho}}{\rho} - 1\right)\mathbf{g}}_{\text{Buoyancy}}$$

Inertia

$$\rho(\overline{p} + p', s) = \quad \overline{\rho}(\Phi) \qquad \simeq \rho_0 = cst$$

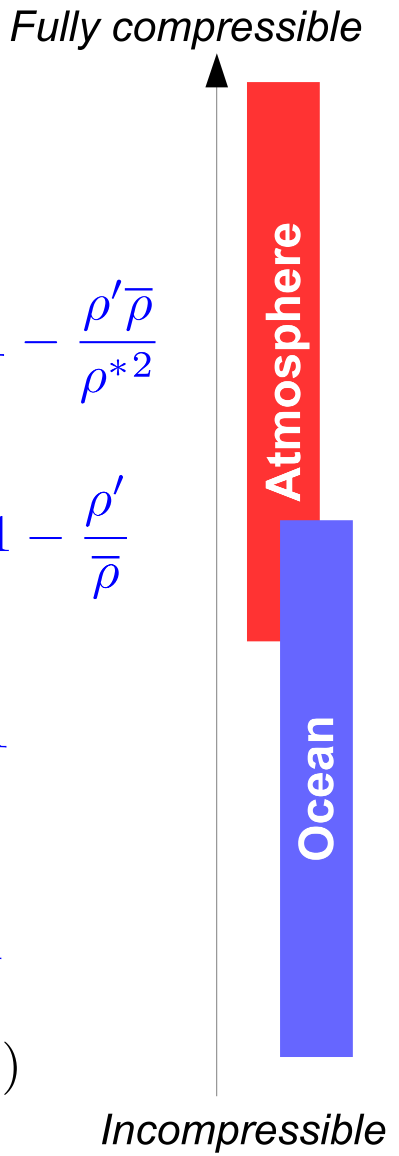
$$+\rho(\overline{p}, s) - \rho(\overline{p}, \overline{s}) \qquad \simeq \rho(p_0, s) - \rho(p_0, \overline{s})$$

$$+\rho(\overline{p} + p', s) - \rho(\overline{p}, s) \qquad \simeq c^{-2}p' \quad \simeq 0$$

$$\boxed{\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p' = -b\mathbf{g}}$$

- Exact : $\rho = \rho(p, s)$ $\quad b = \dfrac{\overline{\rho}}{\rho} - 1$
- Pseudo-incompressible : $\rho = \rho^*(\Phi, s)$ $\quad b = \dfrac{\overline{\rho}}{\rho^*} - 1 - \dfrac{\rho'\overline{\rho}}{\rho^{*2}}$
- Anelastic : $\rho = \overline{\rho}(\Phi)$ $\quad b = \dfrac{\overline{\rho}}{\rho^*} - 1 - \dfrac{\rho'}{\overline{\rho}}$
- Depth-dependent Boussinesq : $\rho = \rho_0 = cst$ $\quad b = \dfrac{\overline{\rho}}{\rho^*} - 1$
- Simple Boussinesq : $\rho = \rho_0 = cst$ $\quad b = \dfrac{\overline{\rho}}{\rho_0^*} - 1$

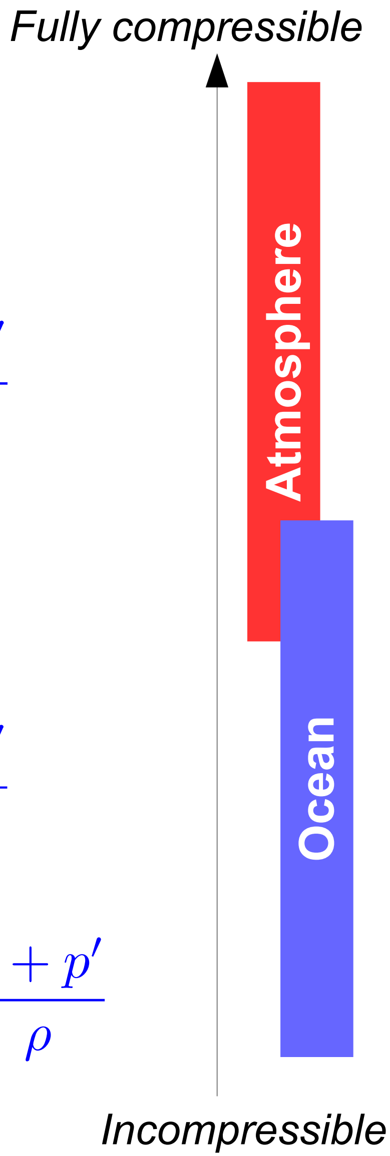
$$\rho_0^*(s) \equiv \rho(p_0, s)$$

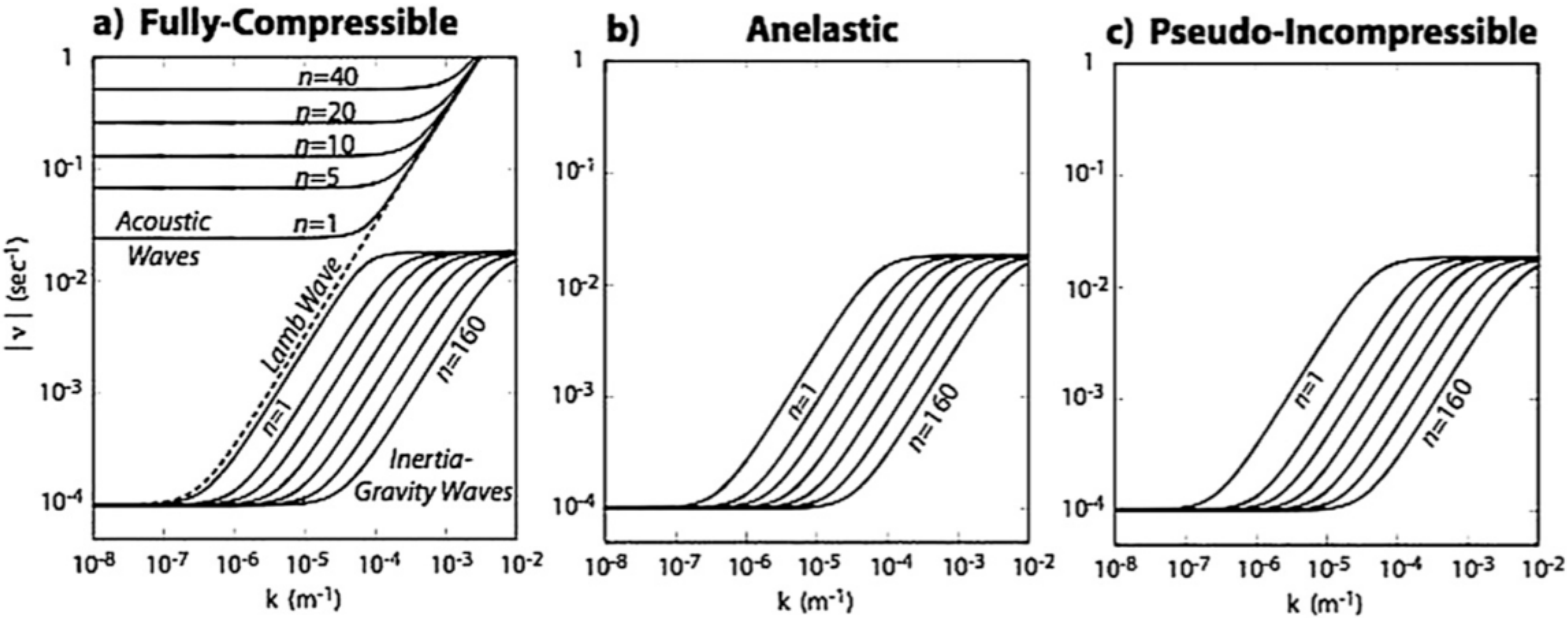


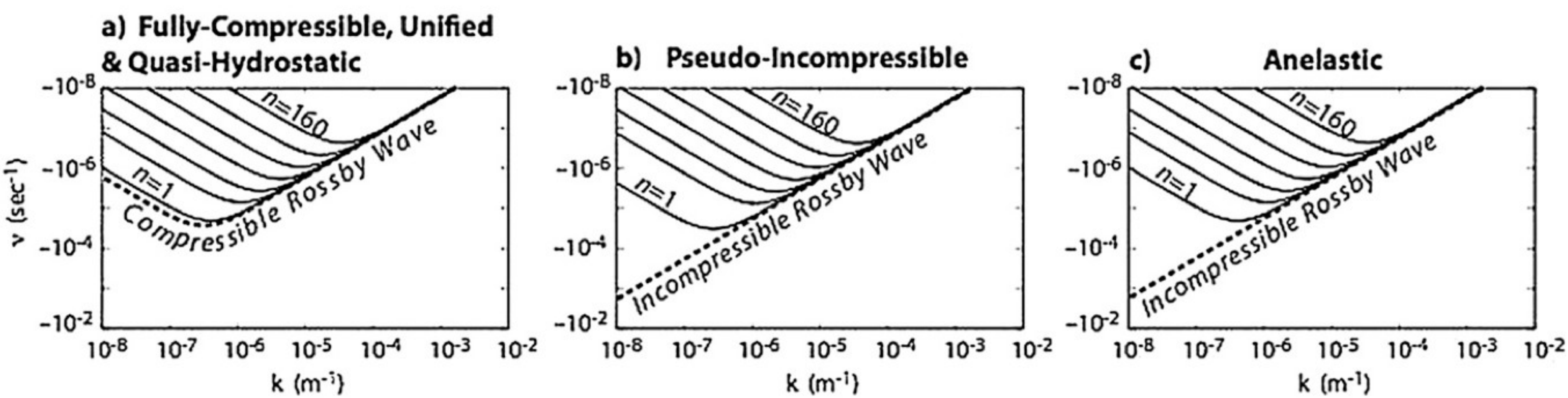
All the above combinations conserve energy/momentum/potential vorticity.

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0 \qquad \frac{\partial L}{\partial p'} = 0 \qquad L = K(\mathbf{x}, \dot{\mathbf{x}}) - E(\mathbf{x}, \rho, s, p')$$

- Exact : $$E = \Phi + h(\overline{p} + p', s) - \frac{\overline{p} + p'}{\rho}$$
- Pseudo-incompressible : $$E = \Phi + h(\overline{p}, s) + \frac{p'}{\rho^*} - \frac{\overline{p} + p'}{\rho}$$
- Anelastic : $$E = \Phi + h(\overline{p}, s) + \frac{p'}{\overline{\rho}} - \frac{\overline{p} + p'}{\rho}$$
- Depth-dependent Boussinesq : $$E = \Phi + h(\overline{p}, s) + \frac{p'}{\rho_0} - \frac{\overline{p} + p'}{\rho}$$
- Simple Boussinesq : $$E = \Phi\left(1 - \frac{\rho_0}{\rho_0^*(s)}\right) + \frac{p'}{\rho_0} - \frac{\overline{p} + p'}{\rho}$$




a) Fully-Compressible
b) Anelastic
c) Pseudo-Incompressible
n=40
n=20
n=10
n=5
n=1
Acoustic Waves
Lamb Wave
n=1
n=160
Inertia-Gravity Waves
|ν| (sec^{-1})
k (m^{-1})
n=1
n=160
n=1
n=160


a) Fully-Compressible, Unified & Quasi-Hydrostatic
b) Pseudo-Incompressible
c) Anelastic
n=160
n=1
Compressible Rossby Wave
ν (sec^{-1})
k (m^{-1})
n=160
n=1
Incompressible Rossby Wave
n=160
n=1
Incompressible Rossby Wave

Pros / cons

All

- Acoustic waves are filtered and do not limit the time step
- A 3D elliptic problem must be solved (hard/expensive)
- Can be combined with the hydrostatic approximation => 1D elliptic problem (easy)

Pseudo-incompressible

- Very accurate, except for long barotropic Rossby waves (too fast)
  => not for global atmospheric models (definition of a reference profile also an issue)
- Elliptic problem has time-dependent coefficients

Anelastic

- Same issues with long barotropic Rossby waves
- Formally, accurate if density close to an adiabatic profile => OK for convection
- In practice, still accurate quite far from neutral stability
  => good for regional nonhydrostatic modelling

Depth-dependent Boussinesq

- Accurate for nearly-incompressible fluid (water)
- Used in most realistic ocean models

Simple Boussinesq

- Not accurate enough for realistic ocean modelling
  But good enough for process studies and idealized modelling
- Conceptual model for atmospheric flow
- Amenable to analytic solutions (with linearized equation of state)

References

Shchepetkin & McWilliams (2011)	Ocean Modeling, **38**(1-2) : 41-70
Tailleux (2012)	doi:10.5402/2012/609701
Eden (2014)	J. Phys. Ocean. **45** : 630-637
Durran (2008)	J. Fluid Mech. **601** : 365–379
Klein & Pauluis (2012)	J. Atmos. Sci **69** : 961-968
Dukowicz (2013)	Mon. Wea. Rev. **141** : 4506
Vasil et al. (2013)	The Astrophysical Journal **773**:169

Hydrostatic modelling

- **What** is the hydrostatic approximation ? When is it **valid** ?
- What are the **degrees of freedom** of a hydrostatic model ?
- How do we **prognose** the degrees of freedom and **diagnose** other quantities ?

answer depends on the choice of *vertical coordinate*

Hydrostatic approximation : basic idea and order-of-magnitude arguments

Vertical momentum balance :

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

If vertical acceleration is small :

$$\frac{\mathrm{d}w}{\mathrm{d}t} \ll g, \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \frac{\partial p}{\partial z} = -\rho g$$

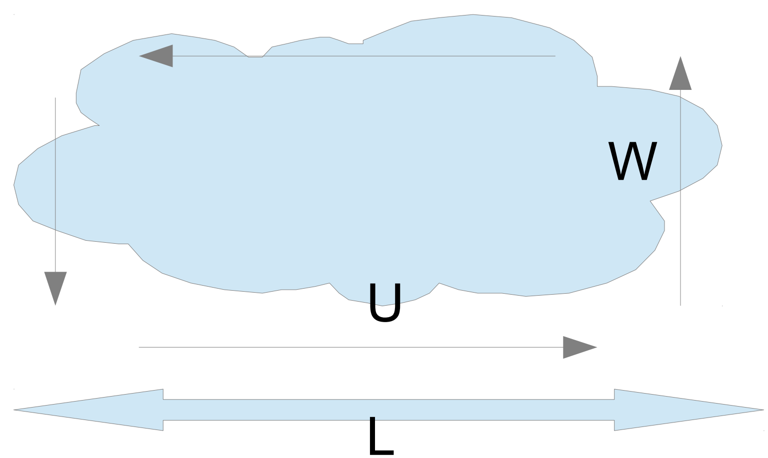
How good is this approximation ?
what we neglect << what is left ?

$$\frac{\mathrm{d}}{\mathrm{d}t} \sim \frac{W}{H} \sim \frac{U}{L} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} \sim \frac{W^2}{H}$$

$$\frac{\partial p}{\partial x} \sim \rho\frac{\mathrm{d}u}{\mathrm{d}t} \sim \rho\frac{U}{L}U$$

$$\frac{\partial p}{\partial z} \sim \frac{L}{H}\frac{\partial p}{\partial x} \sim \frac{\rho U^2}{H}$$

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t}/\rho\frac{\partial p}{\partial z} \sim \left(\frac{W}{U}\right)^2 \sim \left(\frac{H}{L}\right)^2}$$



Atmosphere : OK for large-horizontal-scale circulation (>30km)
KO for convection (storms), orographic flow (mountains).

Ocean : H~1km => OK down to kilometer-scale

Hydrostatic approximation from least action principle

$$\frac{D\mathbf{m}}{Dt} - \frac{\partial L}{\partial \mathbf{x}} - \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) = 0 \qquad \mathbf{m} = \frac{\partial L}{\partial \mathbf{u}} \qquad PV = \frac{1}{\rho}\nabla s \cdot (\nabla \times \mathbf{m})$$

$$L = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \mathbf{R}\cdot\mathbf{u} - \Phi - h(p,s) + \frac{p}{\rho}$$

$$\frac{D\mathbf{u}}{Dt} + \mathrm{curl}\mathbf{R}\times\mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

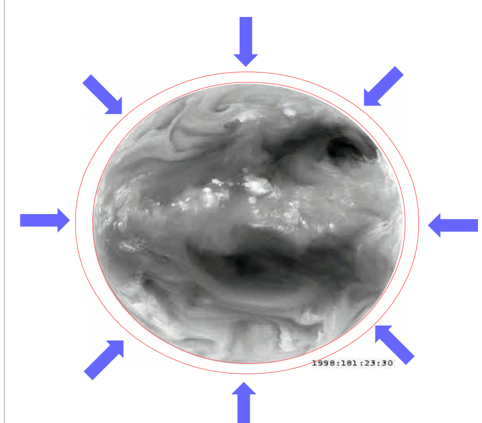
$$E = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + h(p,s) - \frac{p}{\rho} \qquad \mathbf{m} = \mathbf{u} + \mathbf{R}$$

Horizontal projection $\quad \mathbf{u}_H = \mathbf{u} - u_z\mathbf{e}_z$

$$L = \frac{\mathbf{u}_H\cdot\mathbf{u}_H}{2} + \mathbf{R}\cdot\mathbf{u}_H - \Phi - h(p,s) + \frac{p}{\rho}$$

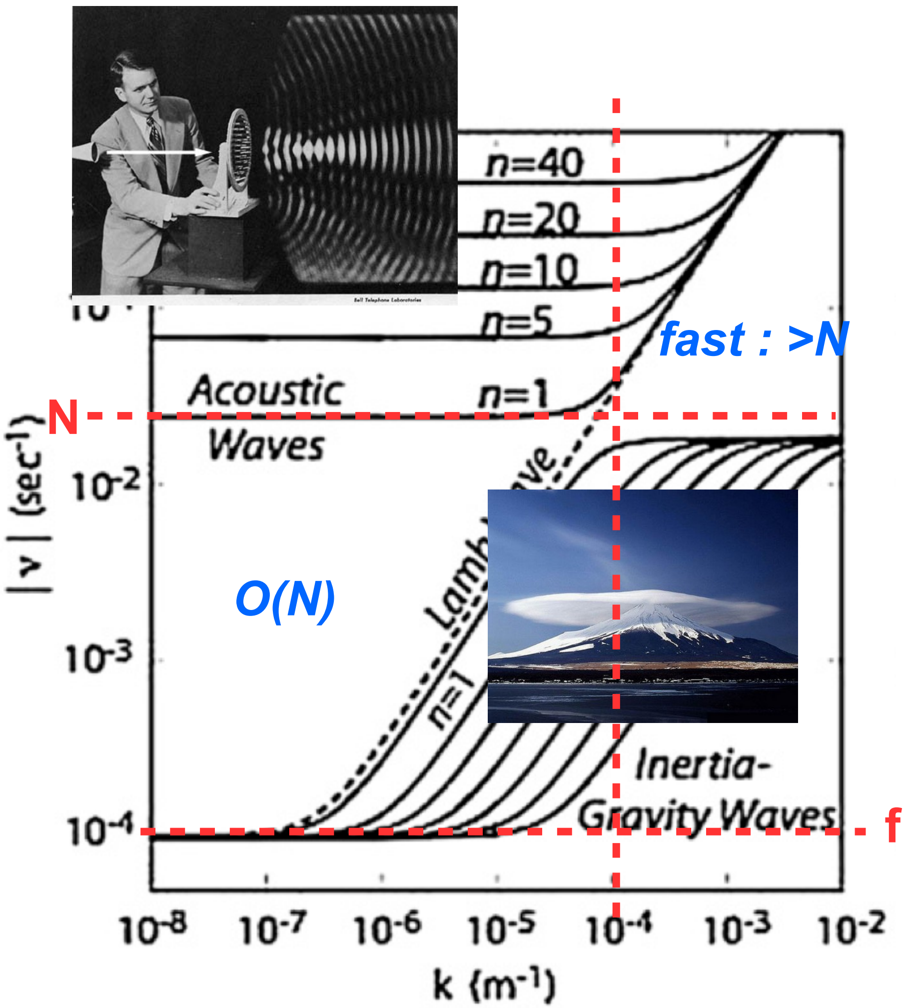
$$\frac{D\mathbf{u}_H}{Dt} + \mathrm{curl}\mathbf{R}\times\mathbf{u}_H + \nabla\Phi + \frac{1}{\rho}\nabla p = 0$$

$$E = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + h(p,s) - \frac{p}{\rho} \qquad \mathbf{m} = \mathbf{u}_H + \mathbf{R}$$

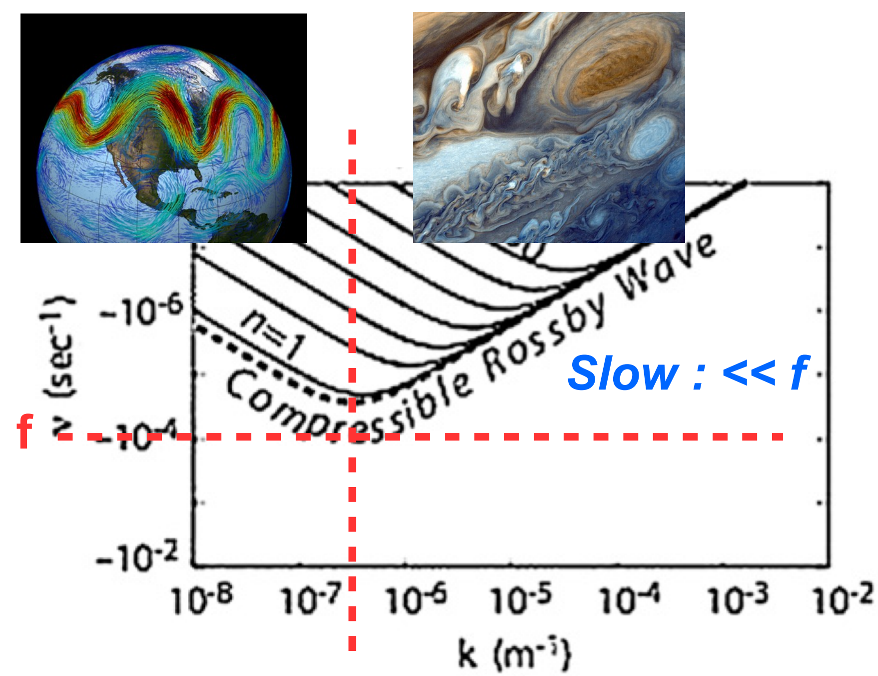


Classes of motion in a gravity-dominated, rotating, compressible flow :

Waves in an isothermal atmosphere at rest



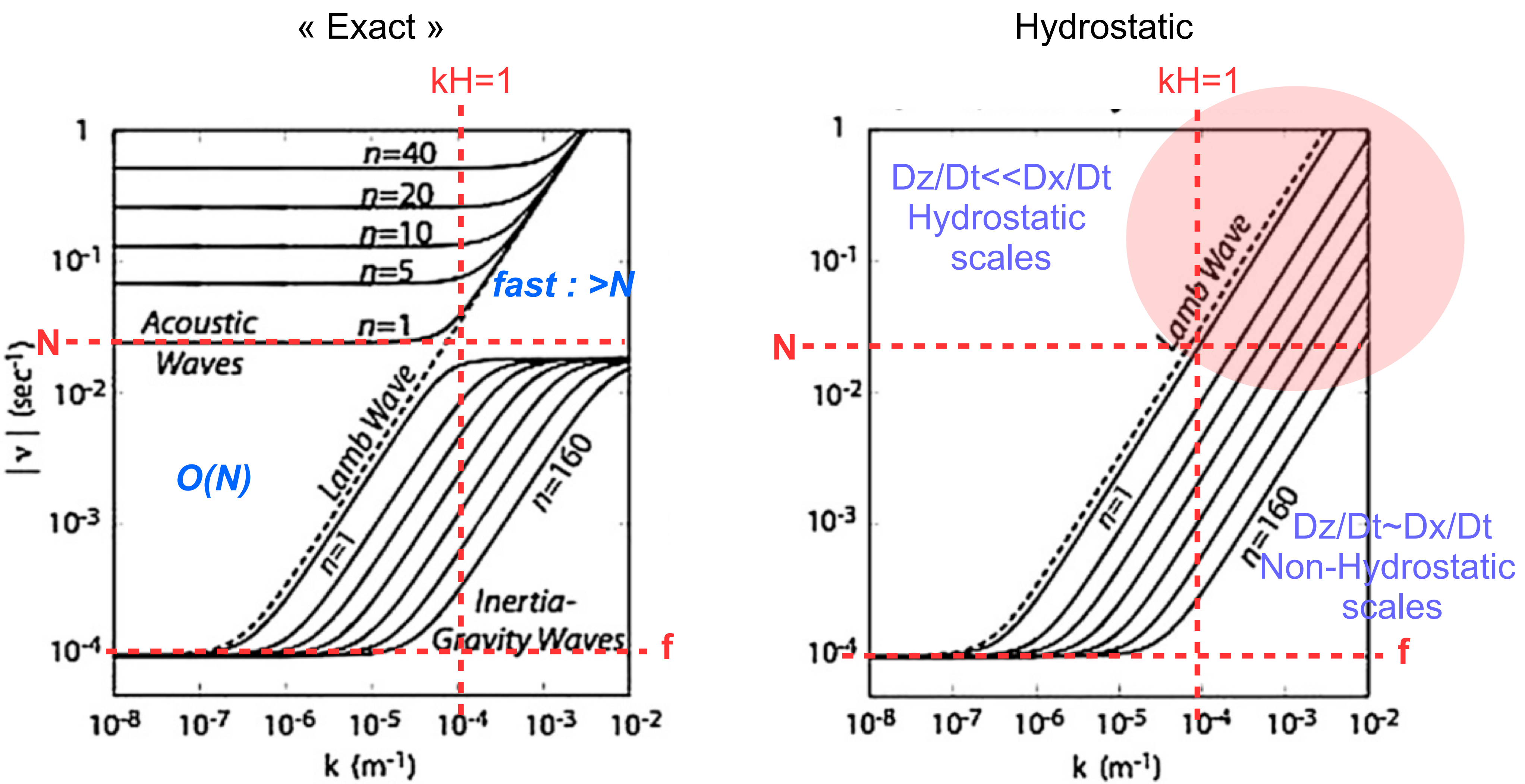
f-plane



beta-plane

Waves in an isothermal atmosphere at rest

Traditional f-plane approximation



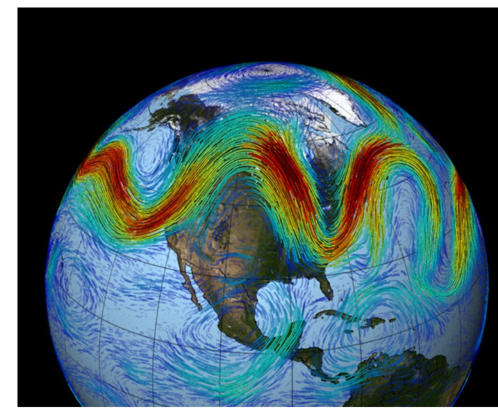
$$\frac{\mathrm{d}w}{\mathrm{d}t} / \rho \frac{\partial p}{\partial z} \sim \left(\frac{W}{U}\right)^2 \sim \left(\frac{H}{L}\right)^2$$

Characteristic scales

- *Velocity* : Sound $c \sim 340 m/s$ Wind $U \sim 30 m/s$
- *Time* : Buoyancy oscillations $N \sim g/c \sim 10^{-2} s^{-1}$ Coriolis $f \sim 10^{-4} s^{-1}$
- *Length* : **Scale height $H=c^2/g=10 km$** **Rossby radius : $R=c/f \sim 1000$ km**

Mach number : $M=U/c \ll 1$

Scale separation : $f/N \sim H/R \ll 1$



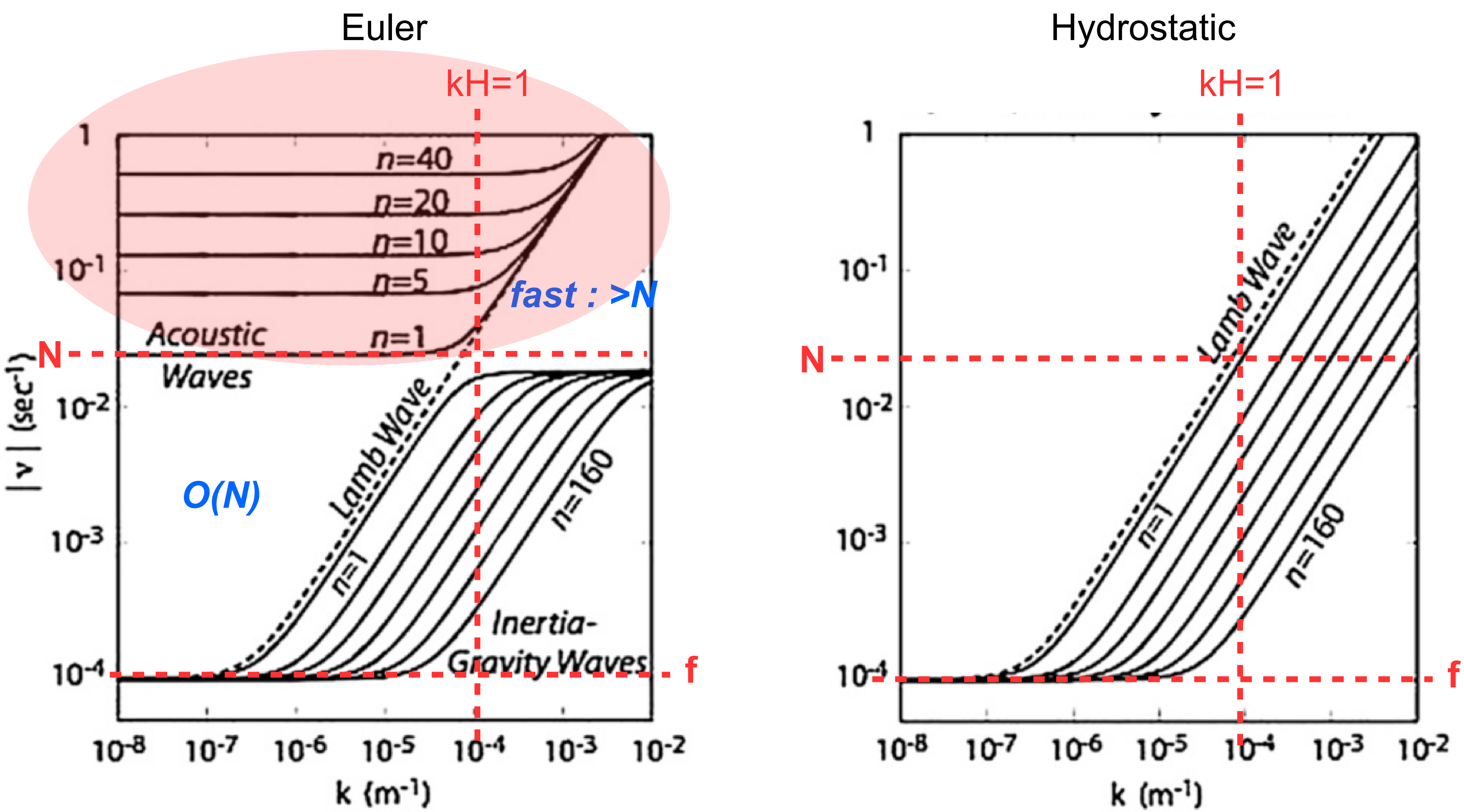
small-scale scale height mesoscale synoptic planetary

1 km 10 km 100 km 1000 km 10000 km

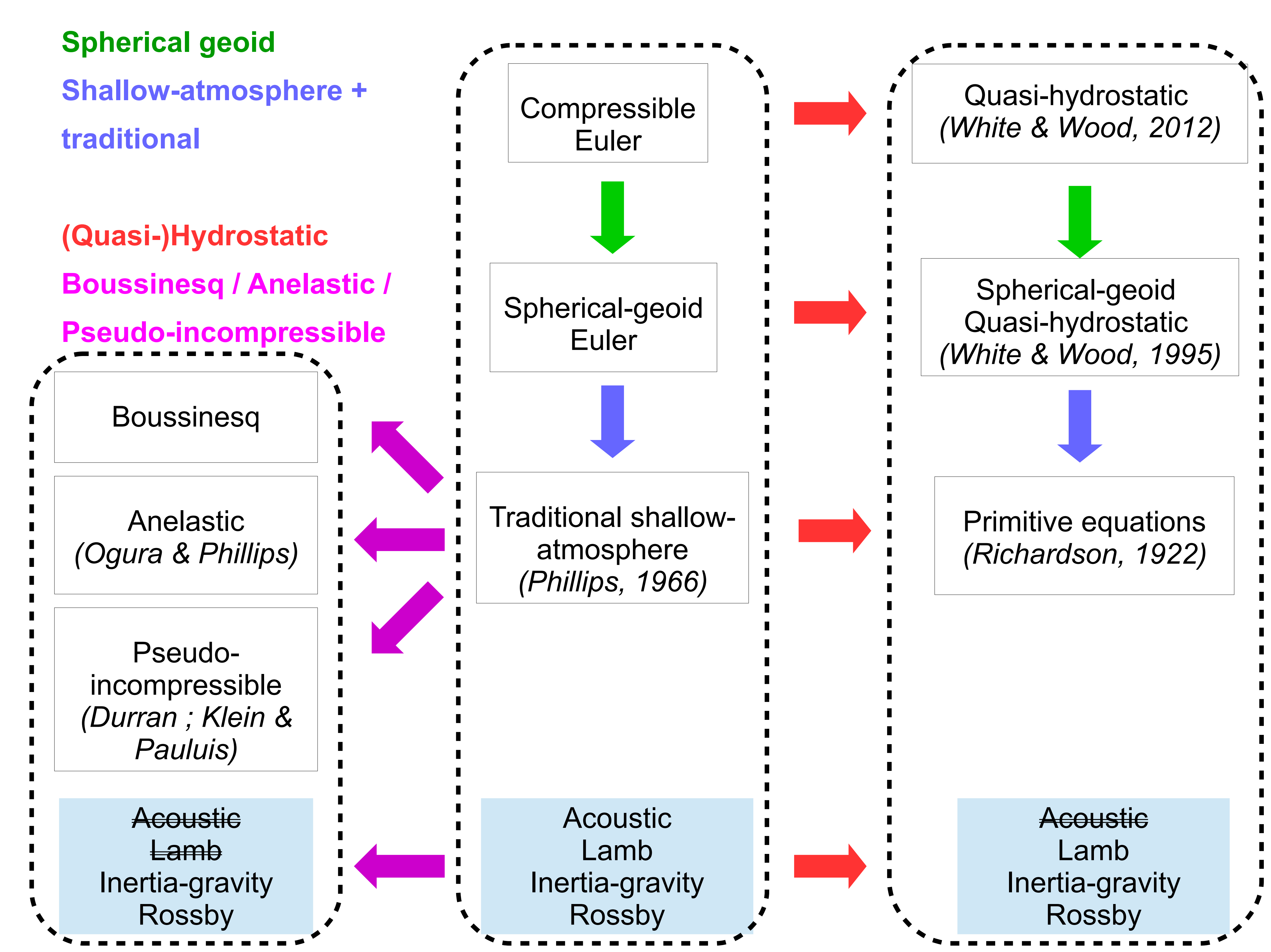
non-hydrostatic **hydrostatic**

Waves in an isothermal atmosphere at rest

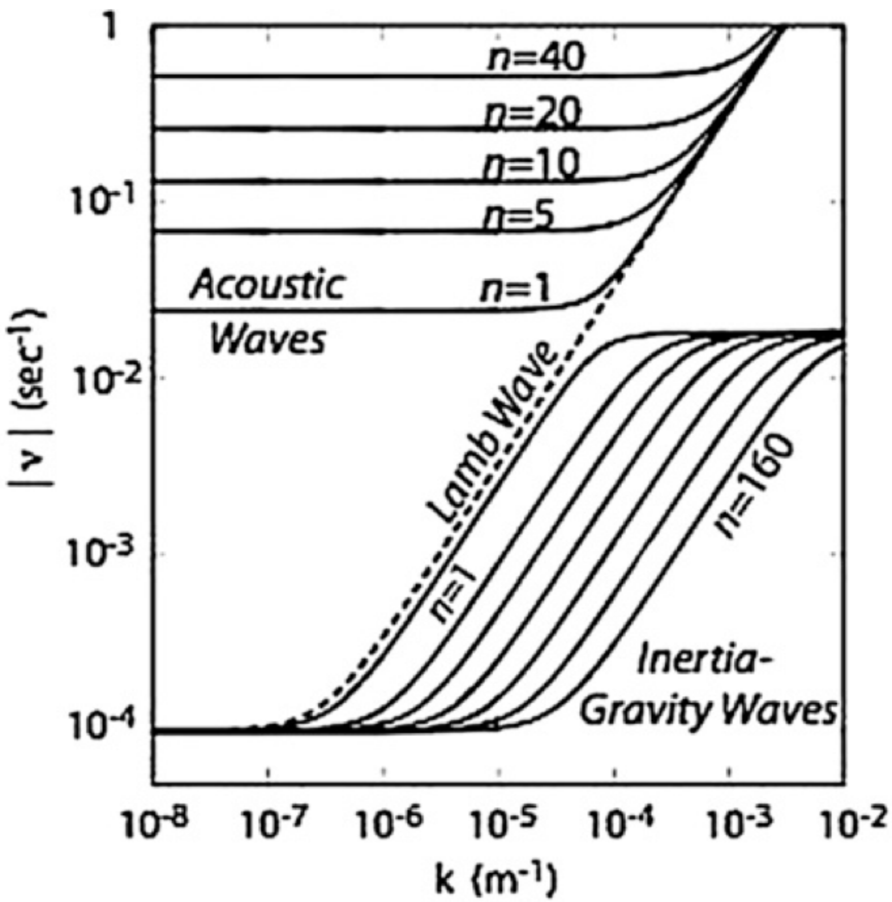
Traditional f-plane approximation

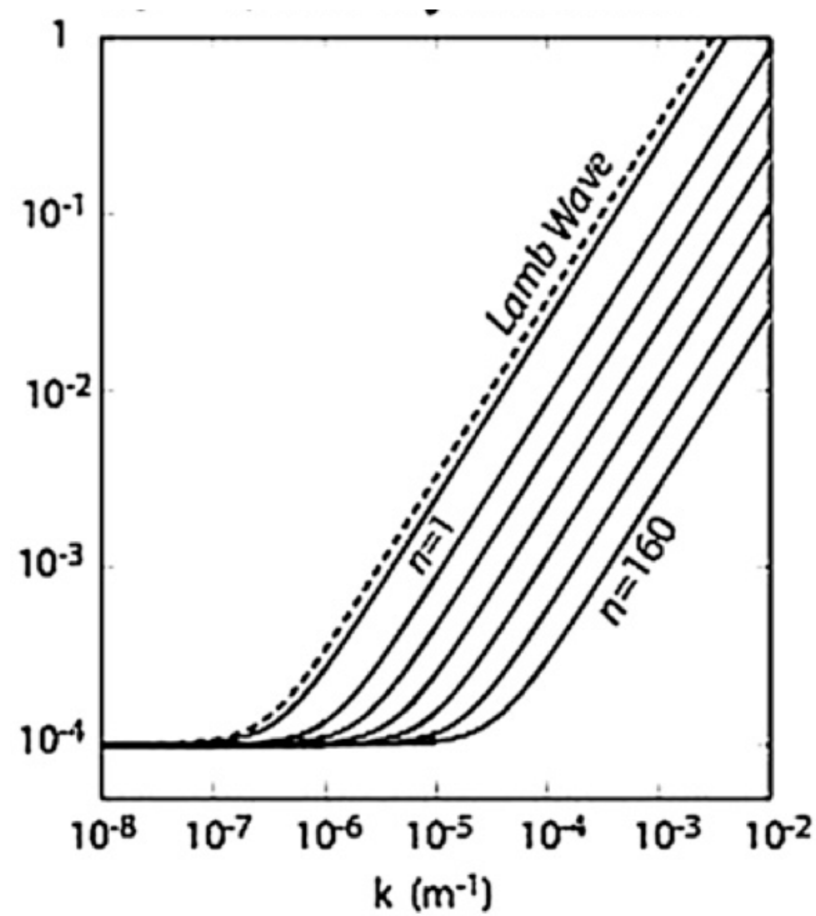


Acoustic waves suppressed : « sound-proof » approximation


Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere
(Phillips, 1966)
Quasi-hydrostatic
(White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic
(White & Wood, 1995)
Primitive equations
(Richardson, 1922)
Boussinesq
Anelastic
(Ogura & Phillips)
Pseudo-incompressible
(Durran ; Klein & Pauluis)
~~Acoustic~~
~~Lamb~~
Inertia-gravity
Rossby
Acoustic
Lamb
Inertia-gravity
Rossby
~~Acoustic~~
Lamb
Inertia-gravity
Rossby

Sound-proofing and degrees of freedom





- Euler equations : 5 prognostic fields (u,v,w, density, entropy)
  => linearized equations lead to a 5x5 determinant
  => degree-5 algebraic equation for ω(k)
  => 5 possible values (roots) for ω(k) :
  2 acoustic, 2 inertia-gravity, 1 Rossby

Hydrostatic :

- only 3 roots : inertia-gravity, Rossby
  => only 3 independent prognostic fields !

- Hydrostatic balance acts as a *constraint* that reduces the numbers of degrees of freedom

- **Which fields have become diagnostic ?**
- **How do I diagnose other fields ?**

Compressible hydrostatic dynamics :

are we ready to prognose ?

$$\begin{aligned}
\partial_z p(\rho, s) + \rho g &= 0 \\
\partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) &= 0 \\
\partial_t s + u\partial_x s + v\partial_y s + w\partial_z s &= 0 \\
\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv + \frac{1}{\rho}\partial_x p &= 0 \\
\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu + \frac{1}{\rho}\partial_y p &= 0
\end{aligned}$$

- Hydrostatic balance yields density given entropy
- 4 prognostic equations for 3 degrees of freedom ??

- There must be an additional diagnostic relationship = *hidden constraint*
- In order to obtain hidden constraints, time-differentiate known constaints (here : hydrostatic balance)

Compressible hydrostatic dynamics : prognostic and diagnostic degrees of feedom

$$\partial_z \left[p\left(\rho, s\right)\right] + \rho g = 0$$

$$\begin{aligned} \partial_t s + w \partial_z s &= \dots \\ \partial_t \rho + \partial_z \left(\rho w\right) &= \dots \end{aligned}$$

$$\partial_z \rho \frac{\partial p}{\partial \rho} + \partial_z s \frac{\partial p}{\partial s} + g \partial_z \rho = 0$$

$$\partial_z \left(\partial_z \left(\rho w\right) \frac{\partial p}{\partial \rho} + w \partial_z s \frac{\partial p}{\partial s} \right) + g \partial_z \left(\rho w\right) = \dots$$

speed of sound

$$\frac{\partial p}{\partial \rho} = c^2$$

$$\boxed{\partial_z \left(\rho c^2 \partial_z w \right) = \dots}$$

Time-differentiating hydrostatic balance
=> diagnostic equation for vertical velocity

- Consistent with the absence of a prognostic equation for vertical velocity
- First obtained by Richardson (1922) but using pressure as a prognostic variable
- Ooyama (1990) : prognosing entropy => « neater form » of Richardson's equation
- Dubos & Tort (2014) : General form for quasi-hydrostatic systems in curvilinear coordinates

Mathematical structure of Richardson's equation

$$\partial_z\left[p\left(\rho,s\right)\right]+\rho g=0$$

$$\begin{aligned}\partial_t s+w\partial_z s &= \ldots\\ \partial_t\rho+\partial_z\left(\rho w\right) &= \ldots\end{aligned}$$

speed of sound

$$\frac{\partial p}{\partial\rho}=c^2$$

$$\partial_z\left(\rho c^2\partial_z w\right)=\ldots$$

- One-dimensional linear problem
- Second-order in z
  requires boundary conditions at top and bottom, e.g. w=0
- *Elliptic* equation for vertical velocity

$$\begin{aligned}\mathcal{F}\left[w\right] &= \frac{1}{2}\int_{z_{bot}}^{z_{top}}\left(X(z)w+\rho c^2\left(\partial_z w\right)^2\right)\mathrm{d}z\\ \delta\mathcal{F} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}\left[w+\varepsilon\delta w\right]\\ &= \int_{z_{bot}}^{z_{top}}\left[X\delta w+\rho c^2\partial_z w\left(\partial_z\delta w\right)\right]\mathrm{d}z\\ &= \int_{z_{bot}}^{z_{top}}\left[X-\partial_z\left(\rho c^2\partial_z w\right)\right]\delta w\mathrm{d}z\end{aligned}$$

$$\partial_z\left(\rho c^2\partial_z w\right)=X\ \Leftrightarrow\ \delta\mathcal{F}=0$$

- Richardson's equation equivalent to minimizing strictly convex functional F
- Unique solution
- Well-posed problem

Compressible hydrostatic dynamics :

are we ready to prognose ?

$$
\begin{aligned}
\partial_z p(\rho, s) + \rho g &= 0 \\
\partial_z \left(\rho c^2 \partial_z w \right) &= \dots \\
\partial_t s + u\partial_x s + v\partial_y s + w\partial_z s &= 0 \\
\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u - fv + \frac{1}{\rho}\partial_x p &= 0 \\
\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v + fu + \frac{1}{\rho}\partial_y p &= 0
\end{aligned}
$$

- Hydrostatic balance yields density given entropy
- Richardson's equation yields vertical velocity
- 3 prognostic equations for 3 degrees of freedom

- Typical compressible hydrostatic models do not work this way
- The reason can be found by having a closer look at *hydrostatic adjustment*

Hydrostatic adjustment

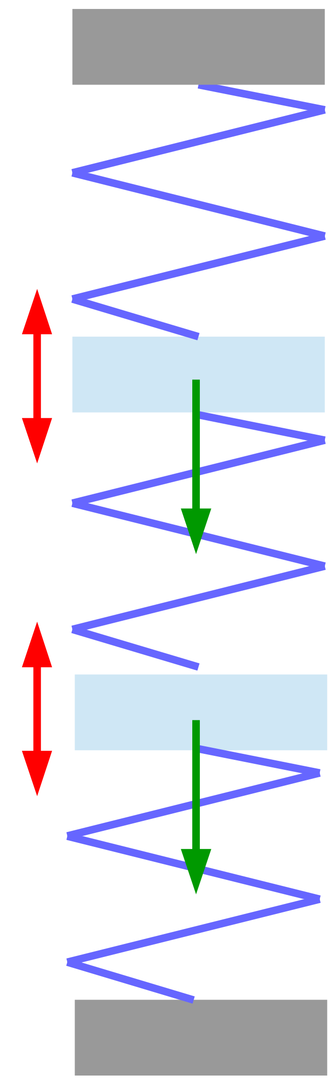
or : how nature imposes hydrostatic balance

Consider

a horizontally homogeneous atmosphere

initial profile ρ(z), s(z) not hydrostatically balanced

what happens ?



Mechanical analogy :

a vertical stack of masses subject to gravity

and coupled by springs

- Vertical forces initially unbalanced
- Vertical acceleration, vertical displacements
- Oscillations / *acoustic* waves
- Until a balance is reached eventually
- Balanced state minimizes *mechanical energy*

Hydrostatic adjustment

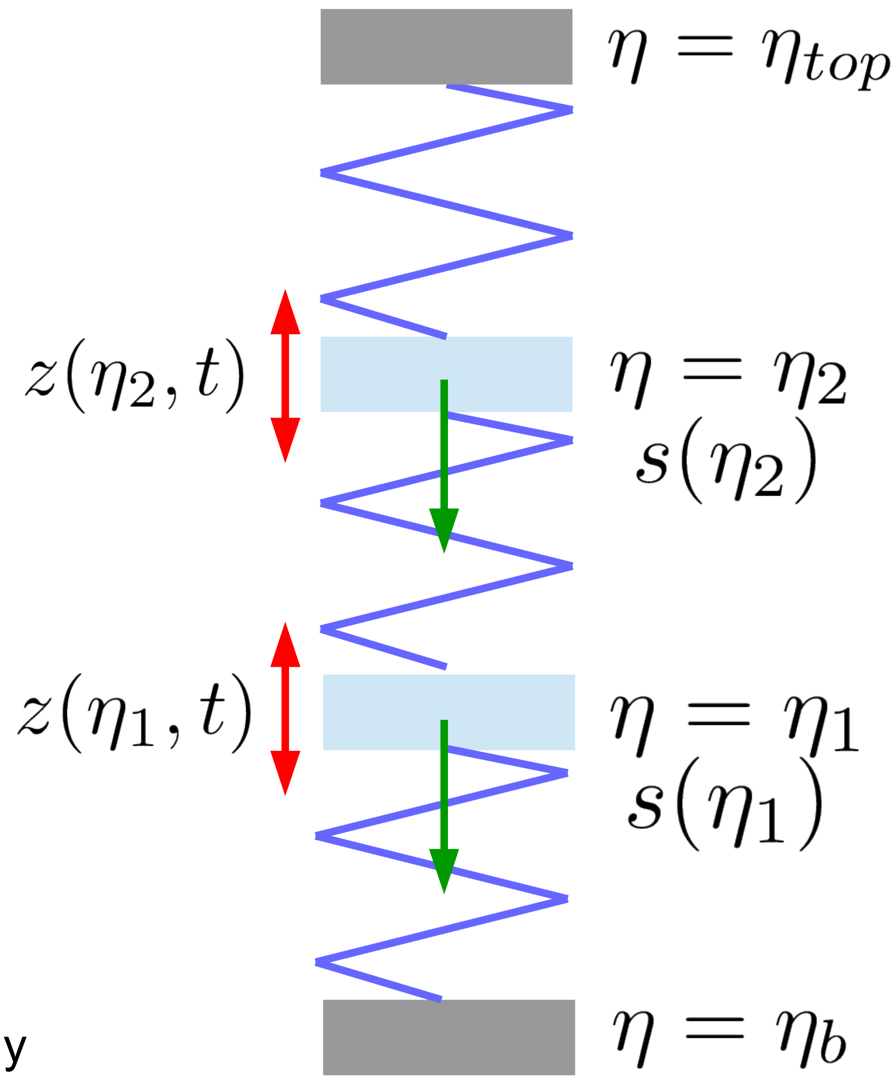
or : how nature imposes hydrostatic balance

$$\mathrm{d}m = \mu \mathrm{d}\eta = \rho \mathrm{d}z$$

$$\mathcal{P}\left[z\right] = \int_{\eta_b}^{\eta_{top}} \left[gz + e\left(\frac{\partial_\eta z}{\mu(\eta)}, \, s(\eta) \right) \right] \mu \mathrm{d}\eta$$

only z can vary here because of conservation of mass and entropy

$$\begin{aligned} \delta\mathcal{P} &= \int_{\eta_b}^{\eta_{top}} \left[g\delta z - \frac{p}{\mu}\partial_\eta \delta z \right] \mu \mathrm{d}\eta \\ &= \int_{\eta_b}^{\eta_{top}} \left[\mu g \delta z + \partial_\eta p \right] \delta z \mathrm{d}\eta \\ &= \int_{z_b}^{z_{top}} \underbrace{\left[\rho g + \partial_z p \right]}_{\text{balance}} \delta z \mathrm{d}z \end{aligned}$$

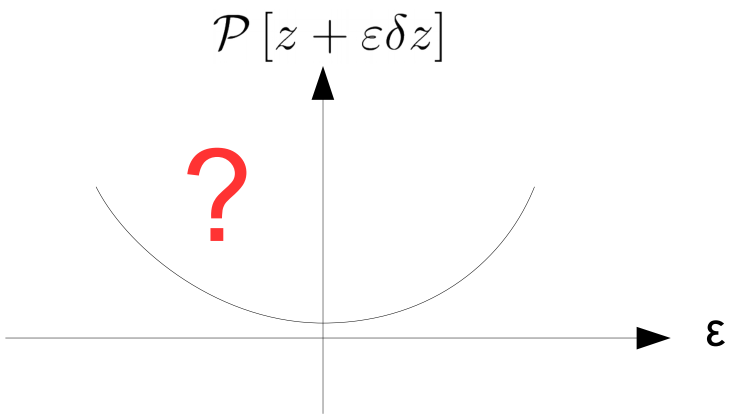


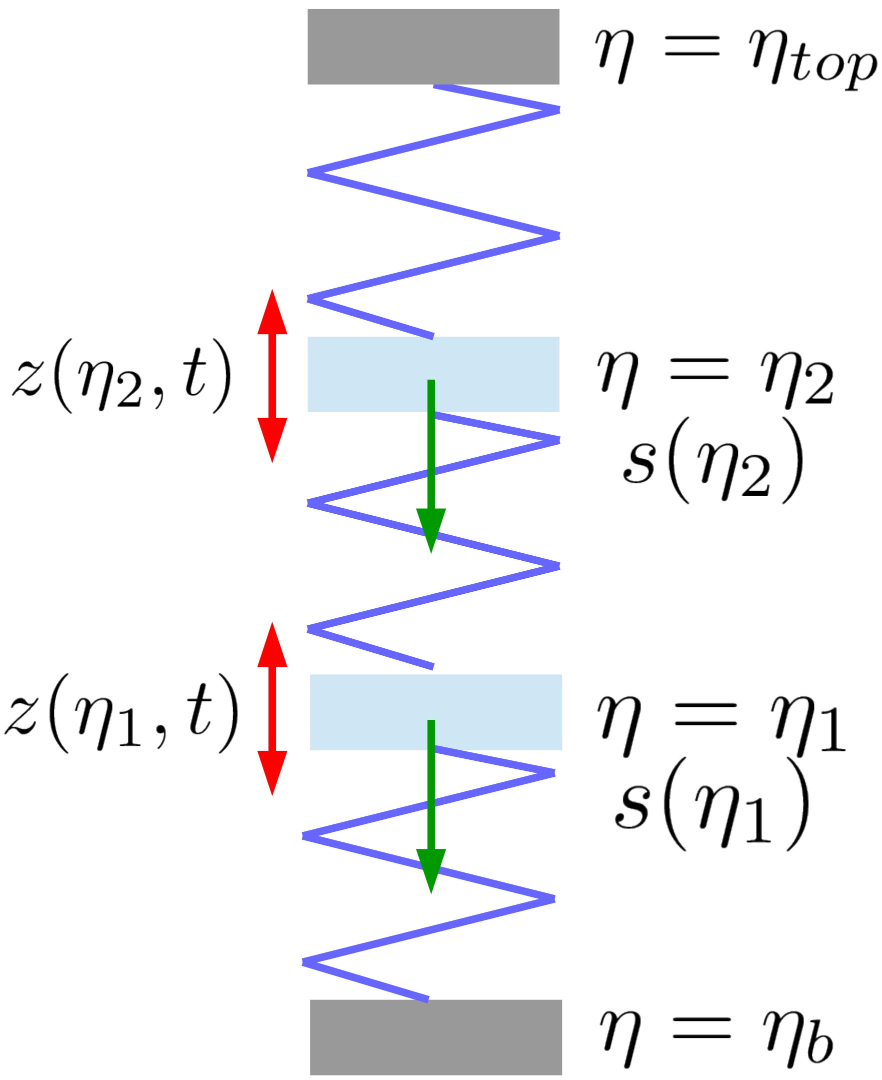
- Air parcels move vertically and adiabatically until hydrostatic balance is restored
- Balanced state minimizes potential+internal energy

Really a *minimum* ?

Hydrostatic adjustment

or : how nature imposes hydrostatic balance

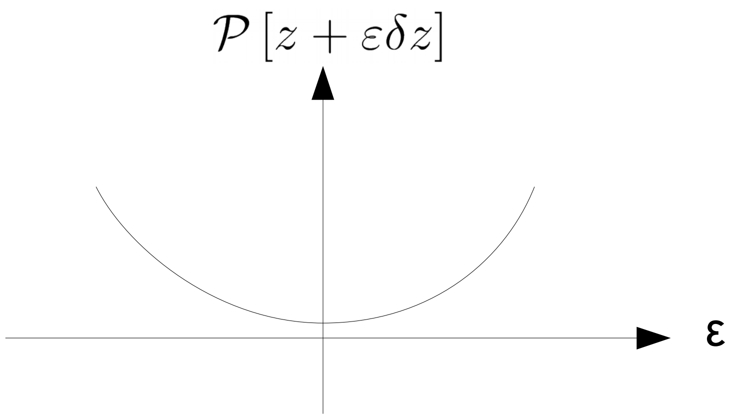


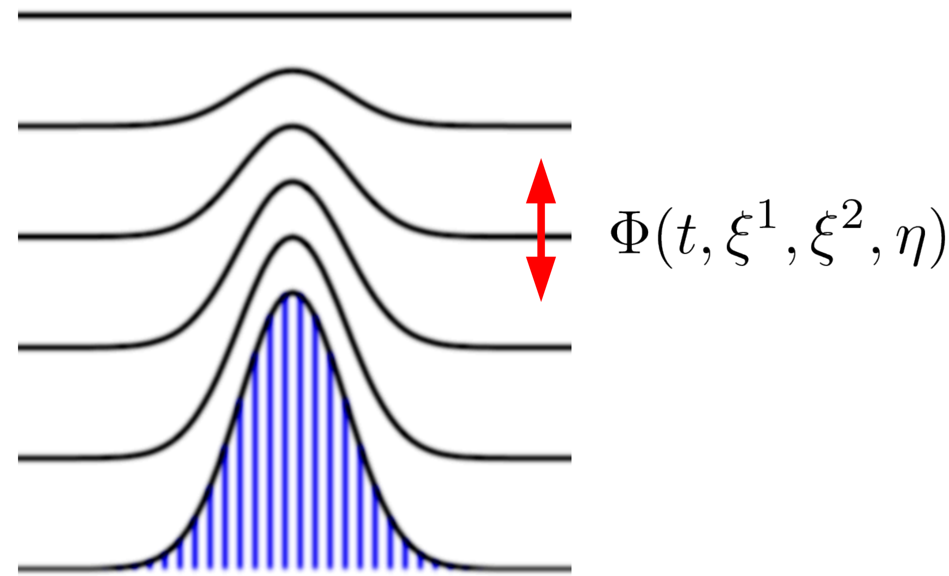


$$\begin{aligned}
\delta^2\mathcal{P} &= \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{P}[z+\varepsilon\delta z] \\
&= \int_{\eta_b}^{\eta_{top}} \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} e\left(\frac{1}{\rho}+\varepsilon\frac{\partial_\eta\delta z}{\mu(\eta)},\, s(\eta)\right)\mu\mathrm{d}\eta \\
&= -\int_{\eta_b}^{\eta_{top}} \partial_\eta\delta z\frac{\mathrm{d}}{\mathrm{d}\varepsilon}p\left(\frac{1}{\rho}+\varepsilon\frac{\partial_\eta\delta z}{\mu(\eta)},\, s(\eta)\right)\mathrm{d}\eta \\
&= \int_{\eta_b}^{\eta_{top}} \left(\frac{\rho}{\mu}\partial_\eta\delta z\right)^2 \mu c^2\mathrm{d}\eta \\
\delta^2\mathcal{P} &= \int_{z_b}^{ztop} (\partial_z\delta z)^2\,\rho c^2\mathrm{d}z
\end{aligned}$$

Really a *minimum.*

Hydrostatic adjustment and vertical coordinate





- Hydrostatic adjustment : 1D, well-posed, nonlinear elliptic problem where *the altitude z of air parcels is the unknown*
- In a hydrostatic model, a hydrostatic adjustment occurs at each time step

suggests to :

- let model layers « float » vertically
- let altitude z be a field rather than coordinate
- Non-Eulerian vertical coordinate

$$\begin{aligned}
\partial_t\mu+\partial_1\left(\mu u^1\right)+\partial_2\left(\mu u^2\right)+\partial_\eta\left(\mu\dot{\eta}\right) &= 0\\
\partial_t s+u^1\partial_1 s+u^2\partial_2 s+\dot{\eta}\partial_\eta s &= \frac{q}{T}
\end{aligned}$$

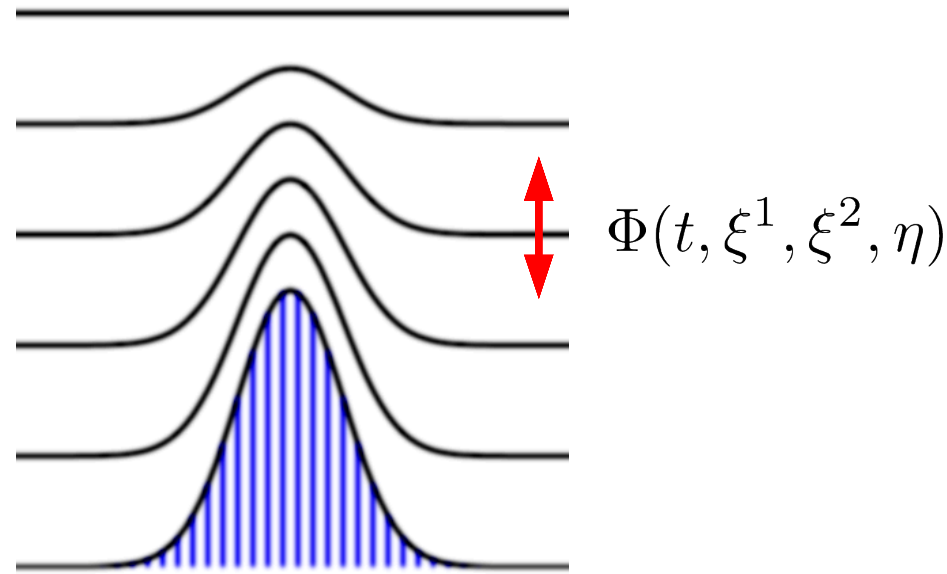
Procedure to *diagnose* $\dot{\eta}$ depends on specific definition of vertical coordinate. This definition is purely *kinematic*.

Generalized vertical coordinates

Isentropic vertical coordinate $s = s(\eta)$

- Entropy becomes *diagnostic*
- Vertical « velocity » given by heating

- Ground usually not an isentrope ...
- Ocean surface usually not an isopycnal...



$$
\begin{aligned}
\partial_t \mu + \partial_1 \left(\mu u^1\right) + \partial_2 \left(\mu u^2\right) + \partial_\eta \left(\mu \dot{\eta}\right) &= 0 \\
\dot{\eta} \frac{\mathrm{d}s}{\mathrm{d}\eta} &= \frac{q}{T}
\end{aligned}
$$

Purely isentropic/isopycnal coordinate not used in practice

Generalized vertical coordinates

Lagrangian vertical coordinate $\dot{\eta} = 0$

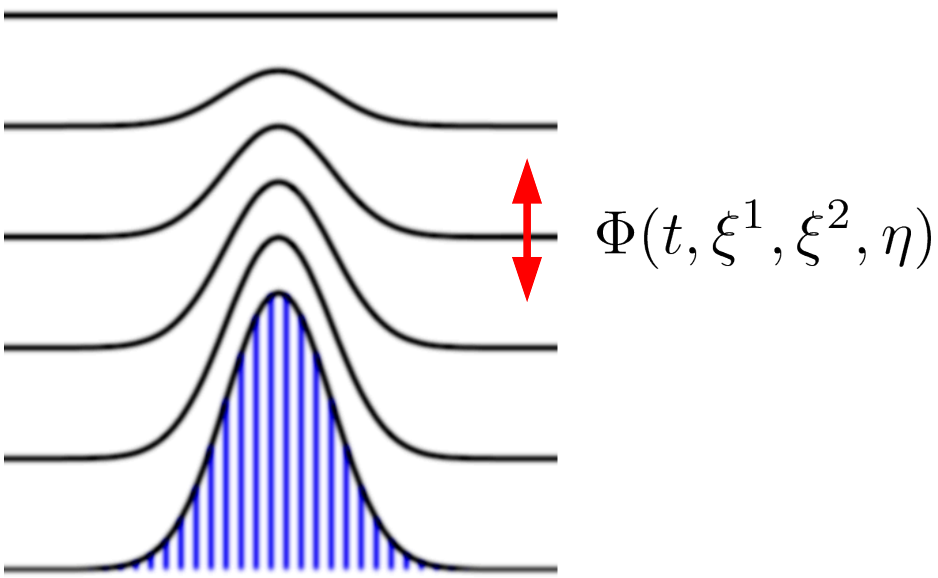
- simple
- initial altitude of layers can be chosen arbitrarily
- layers do not exchange mass, entropy

  => they follow the flow

  => layers tend to *fold* and *cross*

solution :

- exploit arbitraryness of altitude
- *vertical remap* before layers fold/cross



$$\begin{aligned} \partial_t \mu + \partial_1 \left(\mu u^1\right) + \partial_2 \left(\mu u^2\right) &= 0 \\ \partial_t s + u^1 \partial_1 s + u^2 \partial_2 s &= \frac{q}{T} \end{aligned}$$

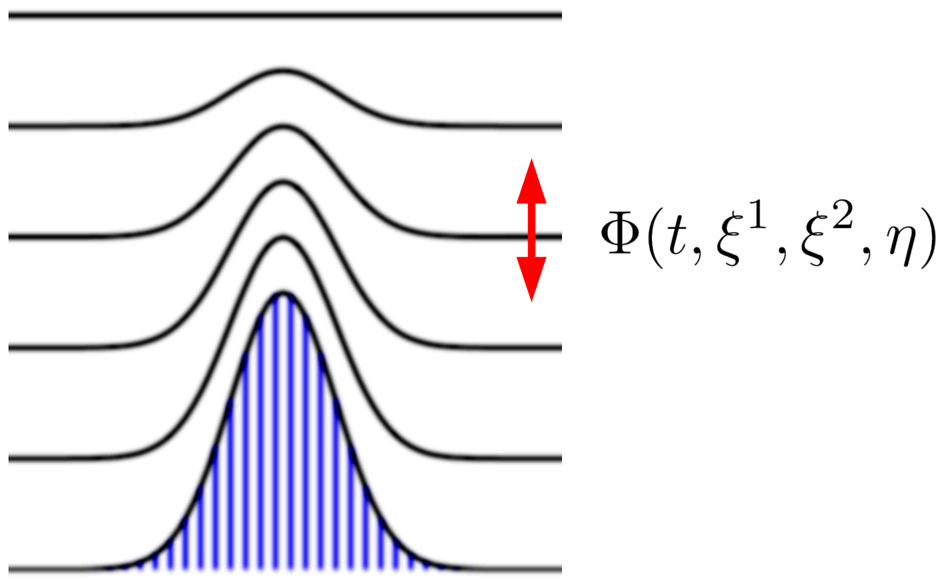
Generalized vertical coordinates

Hybrid mass-based coordinate

- each layer contains a prescribed fraction of the total mass of the column + fixed amount

$$
\begin{aligned}
M(t,\xi^1,\xi^2) &= \int \mu \mathrm{d}\eta \\
\mu &= A(\eta)M + B(\eta)
\end{aligned}
$$

- Layers exchange mass in order to respect this prescription



- *Diagnose mu from total column mass M*
- *Prognose M*
- *Diagnose dmu/dt*
- *Diagnose eta_dot*

$$\partial_t \mu + \partial_1\left(\mu u^1\right) + \partial_2\left(\mu u^2\right) + \partial_\eta\left(\mu\dot\eta\right) = 0$$

$$
\begin{aligned}
\partial_t M + \partial_1 \int \mu u^1 \mathrm{d}\eta + \partial_2 \int \mu u^2 \mathrm{d}\eta &= 0 \\
A(\eta)\partial_t M + \partial_1\left(\mu u^1\right) + \partial_2\left(\mu u^2\right) + \partial_\eta\left(\mu\dot\eta\right) &= 0
\end{aligned}
$$

- There is always some mass in each layer, so layers never fold/cross
- Hybrid coefficient A can be adjusted close to 0 so that upper layers are nearly horizontal

Generalized vertical coordinates & prognostic variables

4+1 fields
Redundant flow description

$$\mu, \; s, \; \Phi$$
+ momentum

Lagrangian

3+1 fields
Non-redundant flow description

$$\mu, \; \Phi$$
+ momentum

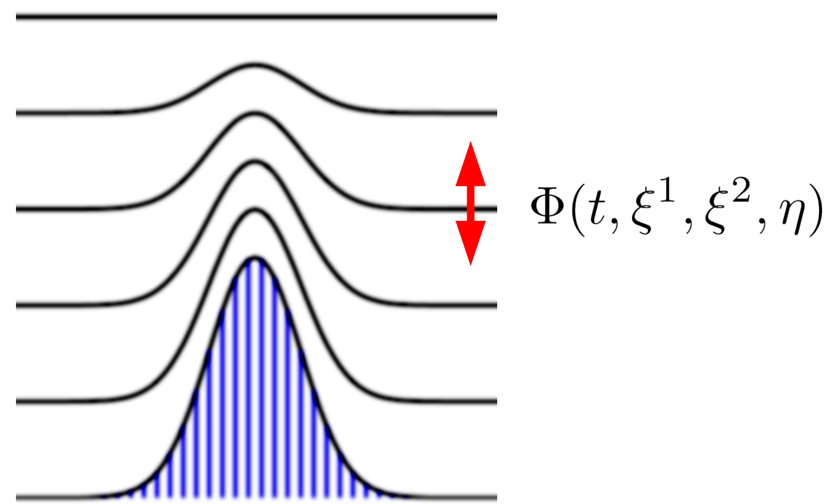
Isentropic / Isopycnal

$$M = \int \mu \mathrm{d}\eta, \; s, \; \Phi$$
+ momentum

Mass-based

Recap : hydrostatic dynamics, generalized vertical coordinates & prognostic variables

- a hydrostatic adjustment occurs at each time step
- altitude z : time-dependent diagnostic field rather than coordinate
- Non-Eulerian vertical coordinate



Hybrid mass-based coordinate

- *Diagnose pseudo-density mu from total column mass M*
- *Prognose M*
- *Diagnose dmu/dt*
- *Diagnose eta_dot*
- *Prognose entropy*

kinematics

- Hydrostatic adjustment => geopotential
- Prognose momentum

dynamics

Lagrangian coordinate

- *Prognose pseudo-density mu*
- *Prognose entropy*
- *If needed, vertical remap*

- Hydrostatic adjustment => geopotential
- Prognose momentum

Additional comments : vertical coordinates, prognostic variables and *non-hydrostatic* modelling

- Generalized vertical coordinates are kinematic : can be used for non-hydrostatic modelling as well (Laprise, 1992)
- Eulerian coordinates more common in non-hydrostatic atmospheric modelling
- One can argue that geopotential and vertical velocity are essentially diagnostic also at non-hydrostatic scales (Dubos & Voitus, 2014)

Non-Eulerian vertical coordinates are a valid choice for non-hydrostatic compressible modelling as well

Generalized vertical coordinates & prognostic variables

Hydrostatic

4+1 fields
Redundant flow description

3+1 fields
Non-redundant flow description

Euler

6 fields
Redundant flow description

5 fields
Non-redundant flow description

$\mu, \; s \text{ or } S = \mu s, \; \Phi$
+ momentum
Lagrangian

$\mu, \; \Phi$
+ momentum
Isentropic / Isopycnal

$\mu = \int M \mathrm{d}\eta, \; s \text{ or } S = \mu s, \; \Phi$
+ momentum
Mass-based

$\mu, \; s \text{ or } S = \mu s$
+ momentum
Eulerian (z-based)

References

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Kasahara (1974)	Mon. Wea. Rev.**102**: 509-522
Ooyama (1990)	J. Atmos. Sci **47**(21) : 2580-2593
Laprise (1992)	Mon. Wea. Rev. **120** : 197-207
Dubos & Tort (2014)	Mon. Wea. Rev. **142**(10) : 3860-3880