Coeurs dynamiques

Formation modélisation numérique de l'océan et de l'atmosphère
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Plan

- Etude de schémas temporels: cas des systèmes
- Diffusion
- Systèmes raides

Cas des systèmes

Schémas d'advection : cas des systèmes

$$\left\{\begin{array}{rcl}\dfrac{\partial \eta}{\partial t}+u_0\dfrac{\partial \eta}{\partial x}+H\dfrac{\partial u}{\partial x} & = & 0\\ \dfrac{\partial u}{\partial t}+u_0\dfrac{\partial u}{\partial x}+g\dfrac{\partial \eta}{\partial x} & = & 0\end{array}\right.$$

Shallow Water
Internal Gravity Waves
Lien avec discrétisation pression (traceurs)-vitesses

$$u(x,t)=\sum_k U_k(t)e^{ikx}$$

$$\eta(x,t)=\sum_k \eta_k(t)e^{ikx}$$

$$U_k(t)=U_k(0)e^{i(kx-\omega t)}, \eta_k(t)=\eta_k(0)e^{i(kx-\omega t)}$$

Relation de dispersion : $\omega_{\pm}=k\left(u_0\pm\sqrt{gH}\right)$

Schémas d'advection : cas des systèmes

$$\left\{\begin{array}{rcl} \dfrac{\partial \eta}{\partial t} + u_0 \dfrac{\partial \eta}{\partial x} + H \dfrac{\partial u}{\partial x} & = & 0 \\ \dfrac{\partial u}{\partial t} + u_0 \dfrac{\partial u}{\partial x} + g \dfrac{\partial \eta}{\partial x} & = & 0 \end{array}\right.$$

$$\frac{\partial y^{\pm}}{\partial t} + \left(u_0 + \sqrt{gH}\right) \frac{\partial y^{\pm}}{\partial x} = 0, \quad y^{\pm} = u \pm \frac{g}{H} \eta$$

Etude du schéma temporel (exemple : Forward-Backward)

$$\left\{\begin{array}{rcl} \dfrac{\eta^{n+1} - \eta^n}{\Delta t} + u_0 \dfrac{\partial \eta^n}{\partial x} + H \dfrac{\partial u^n}{\partial x} & = & 0 \\ \dfrac{u^{n+1} - u^n}{\Delta t} + u_0 \dfrac{\partial u^n}{\partial x} + g \dfrac{\partial \eta^{n+1}}{\partial x} & = & 0 \end{array}\right. \longrightarrow \begin{pmatrix} \eta_k^{n+1} \\ U_k^{n+1} \end{pmatrix} = A \begin{pmatrix} \eta_k^n \\ U_k^n \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ ikg\Delta t & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 - iu_0 k \Delta t & -iHk\Delta t \\ 0 & 1 - iku_0 \Delta t \end{pmatrix}$$

Schémas d'advection : cas des systèmes

$$\begin{pmatrix} \eta_k^{n+1} \\ U_k^{n+1} \end{pmatrix} = A \begin{pmatrix} \eta_k^{n} \\ U_k^{n} \end{pmatrix}$$

Stable si $\|A\|_2 \leq 1$

Von Neumann : étude de la condition $\rho(A) \leq 1$

(Condition nécessaire, équivalent si la matrice est normale $A^\star A = AA^\star$)

Forward Backward (avec $u_0 = 0$)

Valeurs propres de A : $\lambda_\pm = 1 - \frac{1}{2}(\omega\Delta t)^2 \pm i(\omega\Delta t)\sqrt{1 - \frac{1}{4}(\omega\Delta t)^2}$

Si $(\omega\Delta t) \leq 2, \quad |\lambda_\pm| = 1$ Schéma neutre

Schémas semi-lagrangien : cas des systèmes

$$\left\{\begin{array}{rcl} \frac{\partial \eta}{\partial t} + u_0 \frac{\partial \eta}{\partial x} + H \frac{\partial u}{\partial x} & = & 0 \\ \frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} + g \frac{\partial \eta}{\partial x} & = & 0 \end{array}\right.$$

Schéma Eulérien (Forward - Backward) :

$$\left\{\begin{array}{rcl} \frac{\eta^{n+1} - \eta^n}{\Delta t} + u_0 \frac{\partial \eta^n}{\partial x} + H \frac{\partial u^n}{\partial x} & = & 0 \\ \frac{u^{n+1} - u^n}{\Delta t} + u_0 \frac{\partial u^n}{\partial x} + g \frac{\partial \eta^{n+1}}{\partial x} & = & 0 \end{array}\right.$$

Instable si $u_0 > 0$

Schéma Lagrangien :

$$\left\{\begin{array}{rcl} \frac{\eta^{n+1} - \eta_\star}{\Delta t} + H \frac{\partial u^n}{\partial x} & = & 0 \\ \frac{u^{n+1} - u_\star}{\Delta t} + g \frac{\partial \eta^{n+1}}{\partial x} & = & 0 \end{array}\right.$$

$$\eta_\star = \eta^n(x - u_0 \Delta t), u_\star = u^n(x - u_0 \Delta t)$$

CFL : $|(\sqrt{gH})\Delta t| \leq 2$

Schéma Semi-Implicit Semi-Lagrangian (SISL) :

$$\left\{\begin{array}{rcl} \frac{\eta^{n+1} - \eta_\star}{\Delta t} + H \frac{\partial u^{n+1}}{\partial x} & = & 0 \\ \frac{u^{n+1} - u_\star}{\Delta t} + g \frac{\partial \eta^{n+1}}{\partial x} & = & 0 \end{array}\right.$$

Inconditionnellement stable

$$\eta^{n+1} - gH\Delta t^2 \Delta \eta^{n+1} = \eta_\star - H \Delta t \nabla \cdot u_\star$$

Helmholtz

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Equation parabolique et schéma implicite

$$\frac{\partial T}{\partial t} = \mathcal{K}\frac{\partial^2 T}{\partial z^2}$$

$$\frac{T^{n+1} - T^n}{\Delta t} = \mathcal{K}\left[(1-\theta)\frac{\partial^2 T^n}{\partial z^2} + \theta\frac{\partial^2 T^{n+1}}{\partial z^2}\right]$$

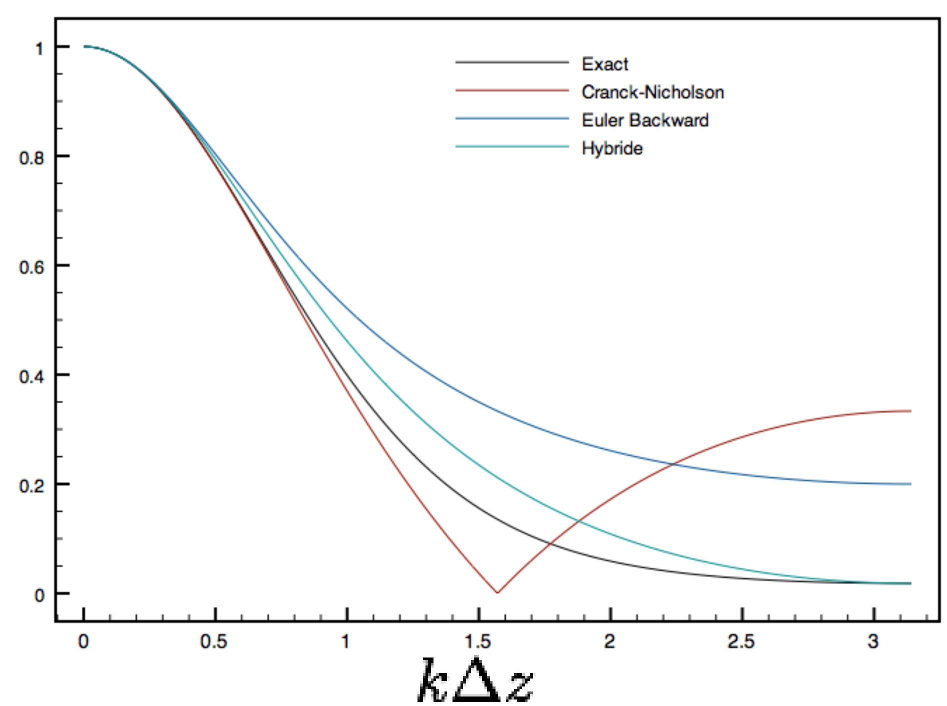
$\theta = \frac{1}{2}$ Crank Nicholson

En explicite ($\theta = 0$), stable si $\mathcal{K}\frac{\Delta t}{\Delta z^2} \leq \frac{1}{2}$

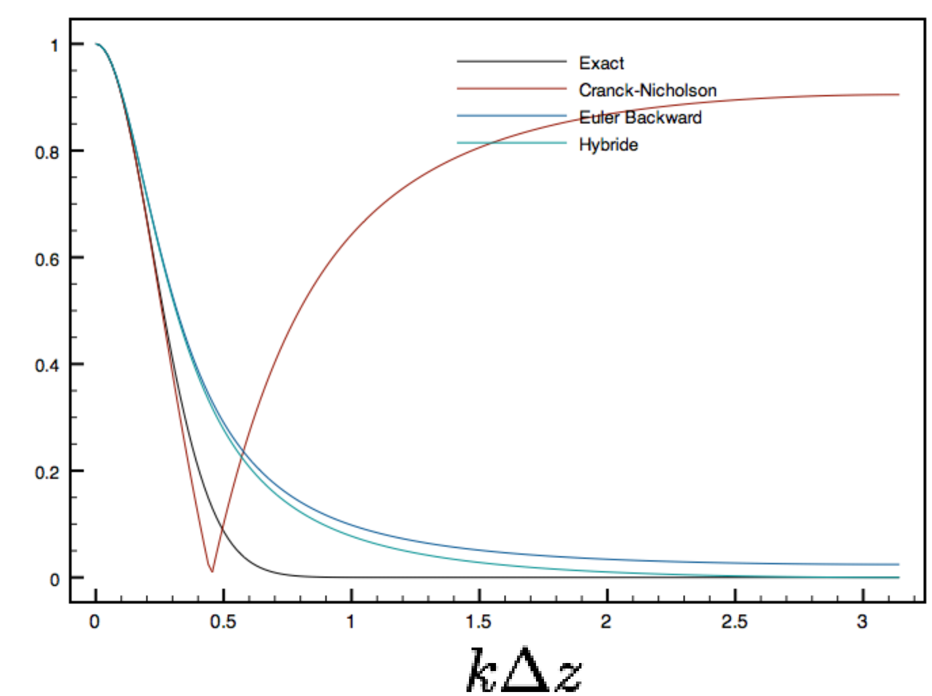
$\theta = 1$ Euler Backward

Amortissement

$\mathcal{K}\frac{\Delta t}{\Delta z^2} = 1$



$\mathcal{K}\frac{\Delta t}{\Delta z^2} = 10$



Equation parabolique et schéma implicite

$\frac{\kappa}{\kappa^{\text{eff}}}$

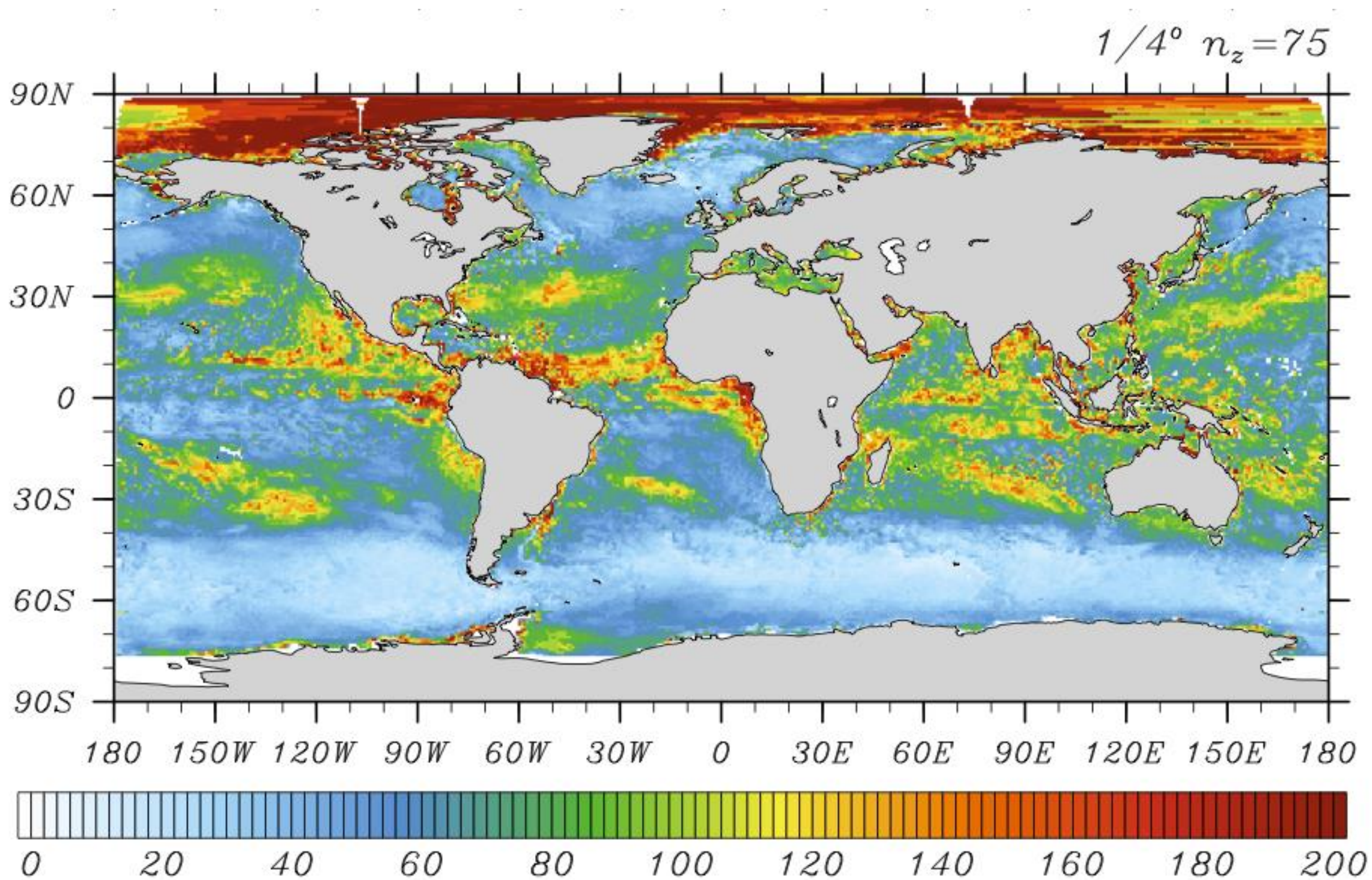


Figure 10: Ratio between the turbulent vertical diffusivity κ and the effective diffusivity κ^{eff} for each water column of the 1/2° (top) and the 1/4° (bottom) configurations. κ^{eff} is the diffusivity in the continuous equation which would give the same damping as the numerical damping (ideally we should get $\kappa/\kappa^{\text{eff}} = 1$). Areas shaded in yellow and red indicate regions with large numerical errors in the computation of vertical diffusion. The value of $\kappa/\kappa^{\text{eff}}$ is computed using (16) with $\sigma^{(2)} = \overline{\sigma}^{(\text{mld})}$ the averaged value of the vertical parabolic Courant number in the mixed layer and $\theta = 2\pi/N_{\text{mld}}$ with N_{mld} the number of grid points in the mixed layer.

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- **Systèmes raides**

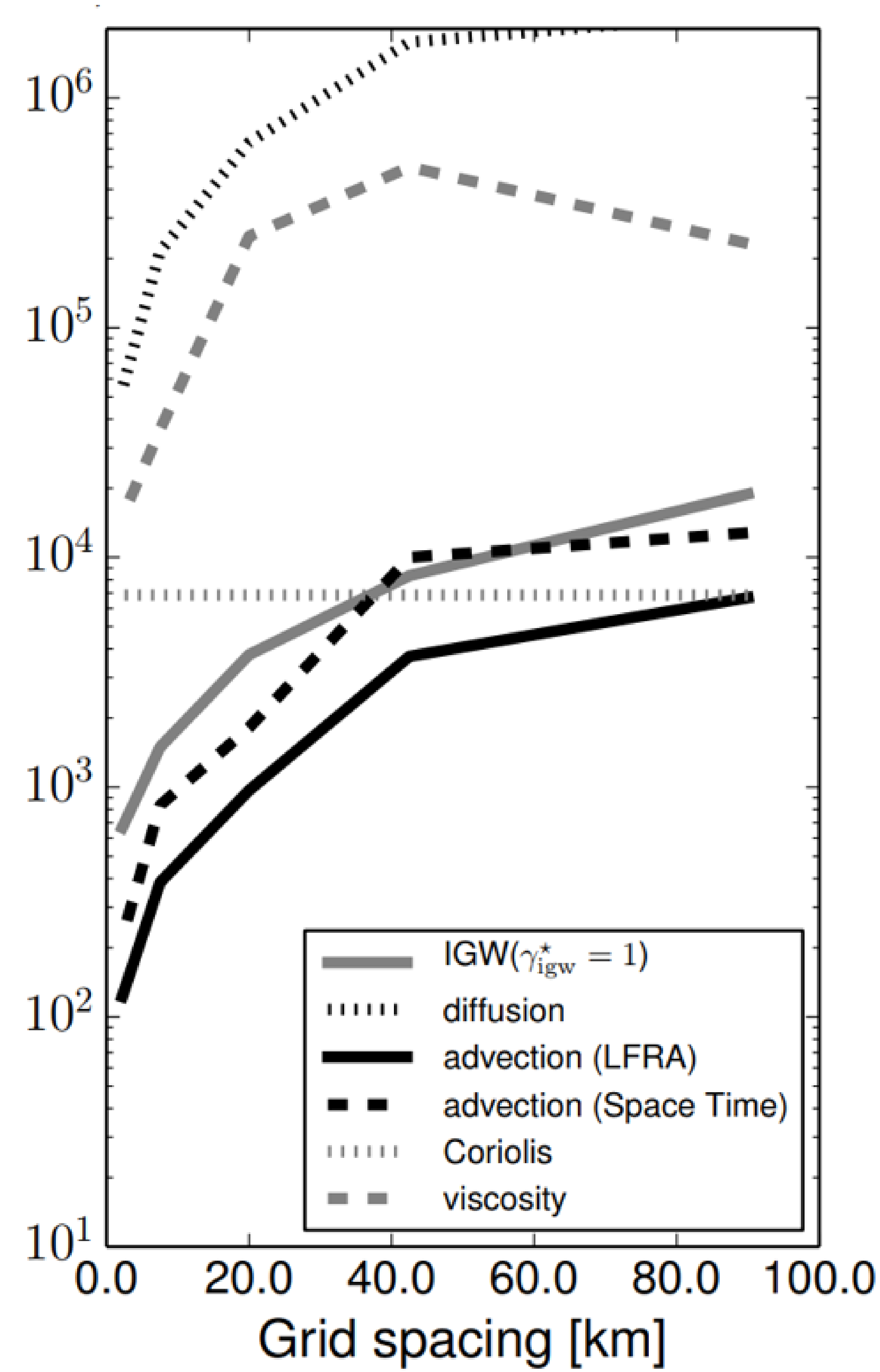
Systèmes raides: Méthodes de splitting

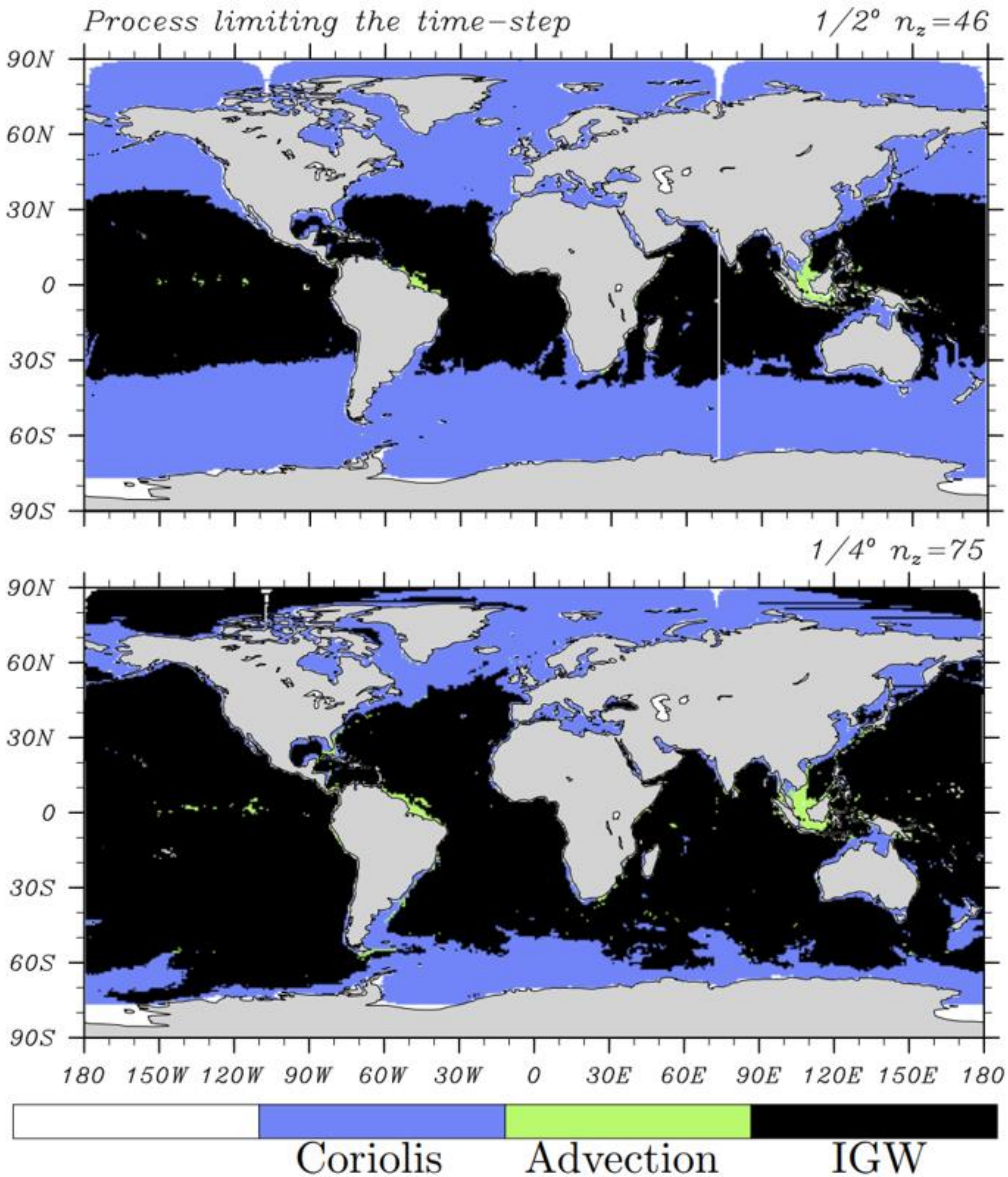
	Ocean	Atmosphere
Acoustic waves c_s	1500	350
External gravity waves $\sqrt{gH}$	200	350
Internal gravity waves NH	3	100
Advection U	0.1	30

OCEAN (Boussinesq)

	Ocean
~~**Acoustic waves**~~ ~~c_s~~	~~1500~~
External gravity waves $\sqrt{gH}$	200
Internal gravity waves NH	3
Advection U	0.1

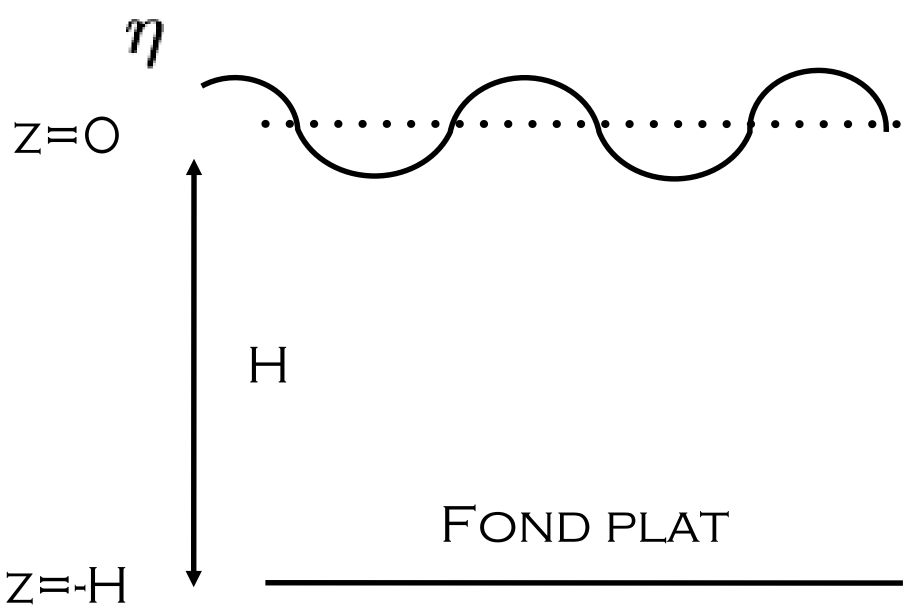
Processus limitant le pas de temps





Process limiting the time−step
1/2° n_z=46
1/4° n_z=75
90N
60N
30N
0
30S
60S
90S
180
150W
120W
90W
60W
30W
0
30E
60E
90E
120E
150E
180
Coriolis
Advection
IGW

1. Linéarisation des équations primitives autour d'un état moyen $(u_0, w_0 = 0, \overline{\rho}(z))$ (2D-vertical)



$$\begin{cases} \partial_t u' + u_0 \partial_x u' + \dfrac{1}{\rho_0}\partial_x p' = 0 \\ \partial_x u' + \partial_z w' = 0 \\ \partial_z p' = -\rho' g \\ \partial_t \rho' + u_0 \partial_x \rho' + w' \partial_z \overline{\rho} = 0 \end{cases} \qquad \begin{cases} \partial_t \eta = w'(0) \\ w'(-H) = 0 \end{cases}$$

2. Définition d'une base modale orthogonale munie d'un produit scalaire.
Sous la forme d'un système aux valeurs propres de type Sturm-Liouville

$$\begin{cases} -\partial_z N^{-2} \partial_z M_q = \lambda_q M_q \\ \dfrac{dM_q}{dz}\Big|_{z=-H} = 0 \\ \dfrac{dM_q}{dz}\Big|_{z=0} = -\dfrac{N^2}{g} M_q(0) \end{cases}$$

Fréquence de Brunt-Väisälä
$$N^2(z) = -\frac{g}{\rho_0}\partial_z \overline{\rho}(z)$$

$$\langle f, g \rangle = \frac{1}{H}\int_H^0 fg\,dz$$

$$\langle M_i, M_j \rangle = \delta_{i,j}$$

$$\longrightarrow (M_q, \lambda_q)$$

3. Décomposition des variables dans la base modale

$$\langle u, M_q\rangle = u_q \qquad \frac{1}{\rho_0}\langle p, M_q\rangle = gh_q$$

$$\sum_q u_q M_q = u \qquad \sum_q g h_q M_q = \frac{1}{\rho_0} p \qquad -\rho_0 \sum_q h_q \partial_z M_q = \rho$$

4. Projection des équations de conservation des moments, de densité et de continuité.
Obtention d'une somme orthogonale de systèmes hyperboliques (caractéristiques)

$$c_q = \frac{1}{\sqrt{\lambda_q}}$$ –> vitesse de phase

$$\left\{\begin{array}{l} \partial_t u_q + u_0 \partial_x u_q + g\partial_x h_q = 0 \\ \partial_t h_q + u_0\partial_x h_q + \dfrac{c_q^2}{g}\partial_x u_q = 0 \end{array}\right. \iff \left\{\begin{array}{l} \partial_t (u_q + \dfrac{g}{c_q} h_q) + (u_0 + c_q)\partial_x (u_q + \dfrac{g}{c_q} h_q) = 0 \\ \partial_t (u_q - \dfrac{g}{c_q} h_q) + (u_0 - c_q)\partial_x (u_q - \dfrac{g}{c_q} h_q) = 0 \end{array}\right.$$

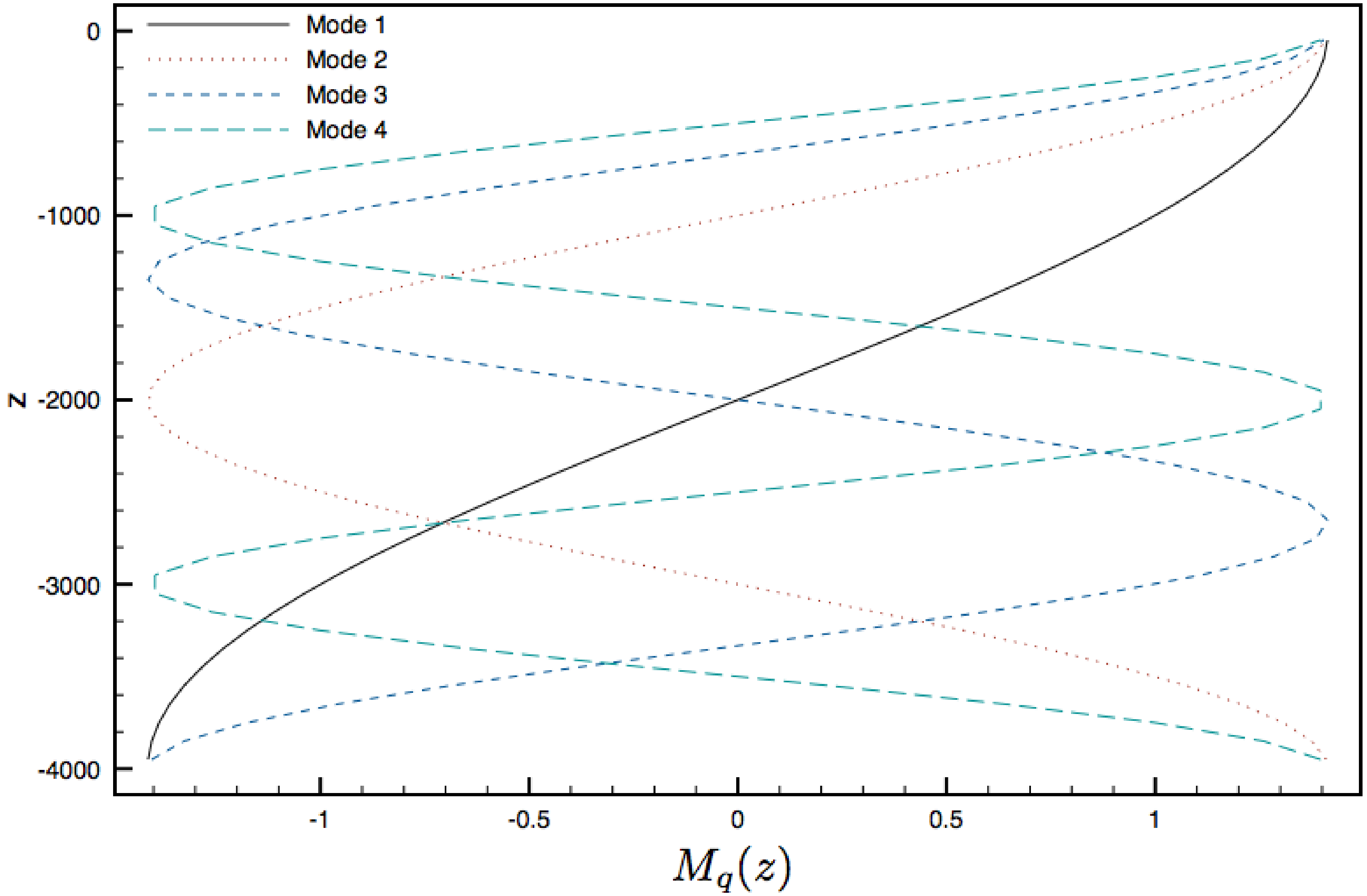
Onde de gravité :
système hyperbolique **couplé vitesse-traceur**
indépendante
vitesse constante : $u_0 \pm c_q$

N constant : une onde externe (barotrope) > des ondes internes (baroclines)

$$c_0 \simeq \sqrt{gH} (\simeq 200 m/s) \gg c_q \simeq \frac{NH}{q\pi} (c_1 \simeq 3 m/s)$$

Mode barotrope quasi-constant sur la verticale.

$$M_0(z) \simeq 1$$



Structure verticale des premiers modes baroclines

Barotropic dynamics and time splitting

- The CFL stability condition on the barotropic mode limits the time step

$$\Delta t_{\mathrm{ext}} < \Delta x / C_{\mathrm{ext}} \text{ where } C_{\mathrm{ext}} = \sqrt{gH} + Ue_{\mathrm{max}}$$

$$H = 4000 \text{ m}, C_{\mathrm{ext}} = 200 \text{ m/s (700 km/h)}, \Delta x = 1\text{km}, \Delta t_{\mathrm{ext}} < 5 \text{ s}$$

- Baroclinic (internal) slow mode: (internal gravity wave phase speed + max advective velocity)

$$C_{\mathrm{in}} \approx 2\text{m/s} + Ui_{\mathrm{max}} \qquad \Delta x = 1\text{km}, \Delta t_{\mathrm{in}} < 8 \text{ mn}$$

- $\Delta t_{\mathrm{in}} / \Delta t_{\mathrm{ext}} \approx 60 - 100!$

Splitting barotrope/barocline

$$c_0 \simeq \sqrt{gH} (\simeq 200 m/s^{-1}) \gg c_q \simeq \frac{NH}{q\pi} (c_1 \simeq 3 m/s^{-1})$$

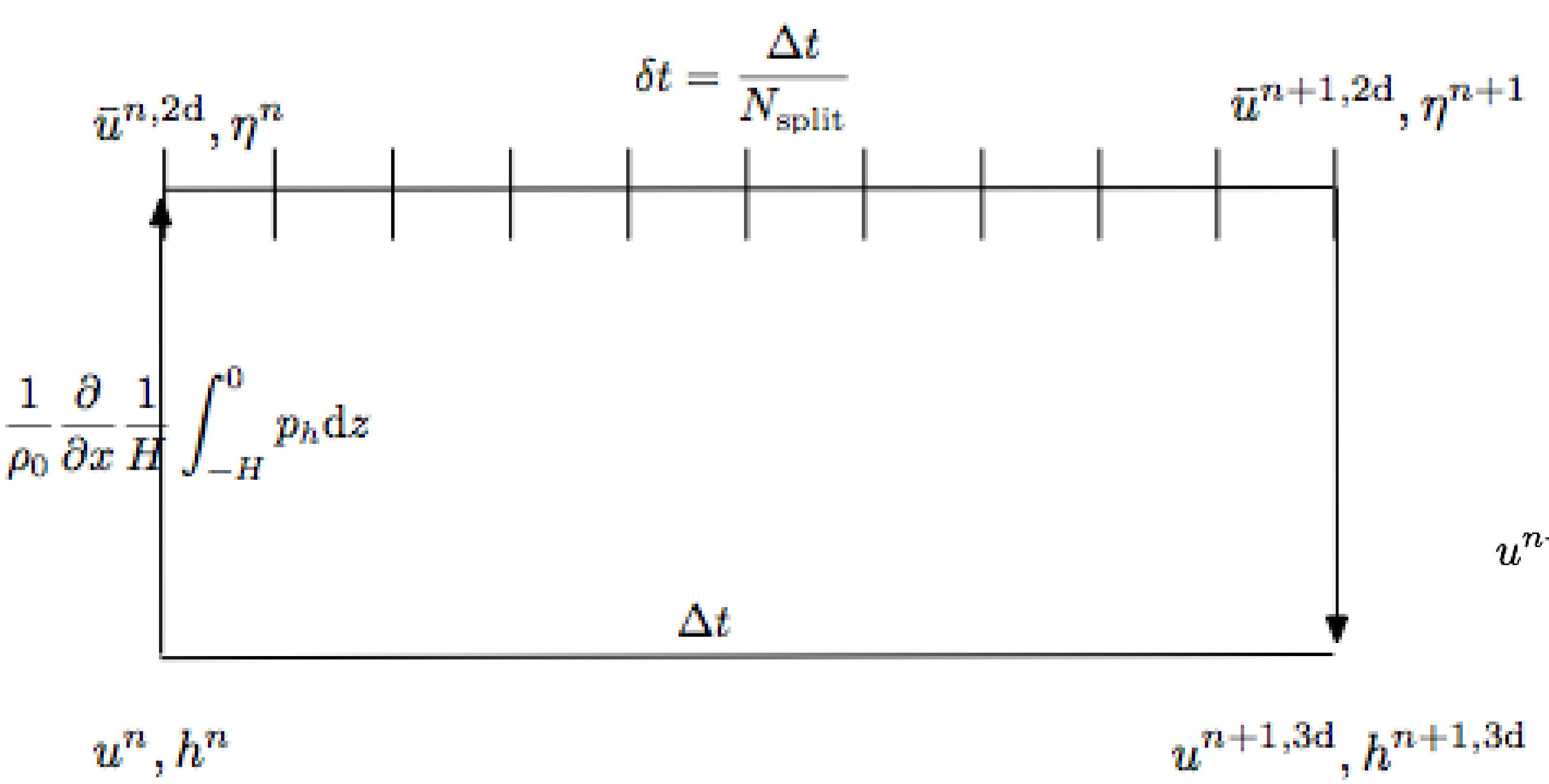
Schéma explicite –> Forte contrainte CFL

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t} + g \dfrac{\partial \eta}{\partial x} & = & -\dfrac{1}{\rho_0} \dfrac{\partial}{\partial x} \dfrac{1}{H} \displaystyle\int_{-H}^{0} p_h \mathrm{d}z \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial H \overline{u}}{\partial x} & = & 0 \end{cases}$$

Système 2D : δt ~ CFL barotrope

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{V} \cdot \nabla) \mathbf{u} + \dfrac{1}{\rho_0} \nabla_h p = 0 \\ \nabla \cdot \mathbf{V} = 0 \\ \partial_z p = -\rho g \\ \partial_t \rho + (\mathbf{V} \cdot \nabla) \rho = 0 \end{cases}$$

Système 3D : Δt ~ CFL barocline



$$u^{n+1,3d,c} = u^{n+1,3d} + \left[\bar{u}^{n+1,2d} - \frac{1}{H} \int_z^0 u^{n+1,3d} \mathrm{d}z \right]$$

Atmosphère

	Ocean	Atmosphere
Acoustic waves c_s	1500	350
External gravity waves $\sqrt{gH}$	200	350
Internal gravity waves NH	3	100
Advection U	0.1	30

Ondes acoustiques et filtrage temporel

- The Courant-Friedrichs-Levy CFL stability condition on the acoustic waves limits the time step: $c_s = 350\mathrm{m.s}^{-1}$
- External - Internal gravity waves : $c_g = 100 - 300\mathrm{m.s}^{-1}$
- Advection : $|U| \leq \frac{c_s}{12}$

$$\left\{ \begin{array}{rcl} \frac{Du_h}{Dt} + \nabla_h P & = & 0 \\ \epsilon_{\mathrm{nh}} \frac{Dw}{Dt} + \frac{\partial P}{\partial z} & = & b \\ \epsilon_{\mathrm{nb}} \frac{DP}{Dt} + c_s^2 \nabla \cdot u & = & 0 \\ \frac{Db}{Dt} & = & 0 \end{array} \right.$$

Filtrage des ondes acoustiques au niveau des équations

$$\begin{cases} \dfrac{Du_h}{Dt} + \nabla_h P & = & 0 \\ \epsilon_{\mathrm{nh}} \dfrac{Dw}{Dt} + \dfrac{\partial P}{\partial z} & = & b \\ \epsilon_{\mathrm{nb}} \dfrac{DP}{Dt} + c_s^2 \nabla \cdot u & = & 0 \\ \dfrac{Db}{Dt} & = & 0 \end{cases}$$

nb NON BOUSSINESQ

nh NON HYDROSTATIQUE

$\epsilon_{\mathrm{nb}=1}$, $\epsilon_{\mathrm{nh}=1}$ PAS DE FILTRAGE

$\epsilon_{\mathrm{nb}=0}$, $\epsilon_{\mathrm{nh}=1}$ FILTRAGE ONDES ACOUSTIQUES (REQUIERT UN SOLVEUR 3D POUR LA PRESSION)

Filtrage des ondes acoustiques au niveau des équations

$\epsilon_{\mathrm{nb}=1},\ \epsilon_{\mathrm{nh}=1}$ Pas de filtrage

Non hydrostatique, compressible

$$\left\{\begin{array}{rcl} \dfrac{Du_h}{Dt} + \nabla_h P & = & 0 \\ \dfrac{Dw}{Dt} + \dfrac{\partial P}{\partial z} & = & b \\ \dfrac{DP}{Dt} + c_s^2 \nabla \cdot u & = & 0 \\ \dfrac{Db}{Dt} & = & 0 \end{array}\right.$$

Traitement numérique : semi-implicite

$$\left\{\begin{array}{rcl} \dfrac{Du_h}{Dt} + \nabla_h P^{n+1} & = & 0 \\ \dfrac{Dw}{Dt} + \dfrac{\partial P^{n+1}}{\partial z} & = & b^n \\ \dfrac{DP}{Dt} + c_s^2 \nabla \cdot u^{n+1} & = & 0 \\ \dfrac{Db}{Dt} & = & 0 \end{array}\right.$$

$\longrightarrow$ $$\Delta P^{n+1} - \frac{1}{(c_s \Delta t)^2} P^{n+1} = G$$ Helmholtz

+ Advection semi-lagrangien $\longrightarrow$ SISL

Traitement numérique : time-splitting

Termes lents

$$m = 1, \ldots, N \qquad \delta t = \frac{\Delta t}{N}$$

$$\left\{ \begin{array}{rcl}
\dfrac{u^{m+1} - u^m}{\delta t} + \dfrac{\partial P^m}{\partial x} & = & -U \dfrac{\partial u^n}{\partial x} - w^n \dfrac{\partial U}{\partial z} \\
\dfrac{w^{m+1} - w^m}{\delta t} + \dfrac{\partial}{\partial z} \left(\dfrac{P^m + P^{m+1}}{2} \right) - b^m & = & -U \dfrac{\partial w^n}{\partial x} \\
\dfrac{b^{m+1} - b^m}{\delta t} + N^2 w^{m+1} & = & -U \dfrac{\partial b^n}{\partial x} \\
\dfrac{P^{m+1} - P^m}{\delta t} + c_s^2 \dfrac{\partial u^{m+1}}{\partial x} + c_s^2 \dfrac{\partial}{\partial z} \left(\dfrac{w^{m+1} + w^m}{2} \right) & = & -U \dfrac{\partial P^n}{\partial x}
\end{array} \right.$$

HEVI : Horizontally Explicit - Vertically Implicit

Références

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