Coordonnées verticales

Formation modélisation numérique de l'océan et de l'atmosphère
Paris, 25 novembre-29 novembre 2019

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Plan

1. **Coordonnees generalisees : formulations des equations**
2. Choix des coordonnées verticales
3. Exemples de coordonnées verticales

Coordonnées verticales généralisées : formulation

$$\left\{\begin{array}{rcl}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+w\dfrac{\partial u}{\partial z} & = & -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z} & = & 0\\ \dfrac{\partial p}{\partial z} & = & -\rho g\\ \dfrac{\partial \rho}{\partial t}+u\dfrac{\partial \rho}{\partial x}+w\dfrac{\partial \rho}{\partial z} & = & 0\end{array}\right.$$

$$s = s(x,y,z,t) \qquad \phi(x,y,z(x,y,s,t),t) = \tilde{\phi}(x,y,s(x,y,z,t),t)$$

$$\left.\frac{\partial \phi}{\partial x}\right|_z = \left.\frac{\partial \tilde{\phi}}{\partial x}\right|_s + \frac{\partial \tilde{\phi}}{\partial s}\left.\frac{\partial s}{\partial x}\right|_z$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \tilde{\phi}}{\partial s}\frac{\partial s}{\partial z}$$

$$\left.\frac{\partial \phi}{\partial t}\right|_z = \left.\frac{\partial \tilde{\phi}}{\partial t}\right|_s + \frac{\partial \tilde{\phi}}{\partial s}\left.\frac{\partial s}{\partial t}\right|_z$$

Coordonnées verticales généralisées : formulation

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} & = & -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} & = & 0 \\
\dfrac{\partial p}{\partial z} & = & -\rho g \\
\dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + w\dfrac{\partial \rho}{\partial z} & = & 0
\end{array}
\right.
$$

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial u}{\partial t} + u\left.\dfrac{\partial u}{\partial x}\right|_s + \dfrac{\partial u}{\partial s}\left(\left.\dfrac{\partial s}{\partial t}\right|_z + u\left.\dfrac{\partial s}{\partial x}\right|_z + w\dfrac{\partial s}{\partial z}\right) & = & -\dfrac{1}{\rho_0}\left[\left.\dfrac{\partial p}{\partial x}\right|_s + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\
\left.\dfrac{\partial u}{\partial x}\right|_s + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial z} & = & 0 \\
\dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial z} & = & -\rho g \\
\dfrac{\partial \rho}{\partial t} + u\left.\dfrac{\partial \rho}{\partial x}\right|_s + \dfrac{\partial \rho}{\partial s}\left(\left.\dfrac{\partial s}{\partial t}\right|_z + u\left.\dfrac{\partial s}{\partial x}\right|_z + w\dfrac{\partial s}{\partial z}\right) & = & 0
\end{array}
\right.
$$

Coordonnées verticales généralisées : formulation

$$\Omega = \left.\frac{\partial s}{\partial t}\right|_z + u \left.\frac{\partial s}{\partial x}\right|_z + w\frac{\partial s}{\partial z}$$

$$\left\{\begin{array}{rcl}
\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \Omega\dfrac{\partial u}{\partial s} & = & -\dfrac{1}{\rho_0}\left[\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\
\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial w}{\partial s}\dfrac{\partial s}{\partial z} & = & 0 \\
\dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial z} & = & -\rho g \\
\dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \Omega\dfrac{\partial \rho}{\partial s} & = & 0
\end{array}\right.$$

Coordonnées verticales généralisées : formulation

$$\left.\frac{\partial z}{\partial x}\right|_z = 0 = \left.\frac{\partial z}{\partial x}\right|_s + \frac{\partial z}{\partial s}\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial z} = 1 = \frac{\partial z}{\partial s}\frac{\partial s}{\partial z}$$

$$\frac{\partial s}{\partial x} = -\left.\frac{\partial z}{\partial x}\right|_s / \frac{\partial z}{\partial s} = -\left.\frac{\partial z}{\partial x}\right|_s / h$$ où $$h = \frac{\partial z}{\partial s}$$

$$\frac{\partial s}{\partial t} = -\left.\frac{\partial z}{\partial t}\right|_s / h$$

$$\Omega = \left.\frac{\partial s}{\partial t}\right|_z + u\left.\frac{\partial s}{\partial x}\right|_z + w\frac{\partial s}{\partial z} \longrightarrow w = h\Omega + \left(\left.\frac{\partial z}{\partial t}\right|_s + u\left.\frac{\partial z}{\partial x}\right|_s\right)$$

Coordonnées verticales généralisées : formulation

Equation de continuité

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial z} = 0$$

$$h\frac{\partial u}{\partial x} + h\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial s} = 0$$

$$h\frac{\partial u}{\partial x} + h\frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial h\Omega}{\partial s} + \frac{\partial}{\partial s}\left(\left.\frac{\partial z}{\partial t}\right|_s + u \left.\frac{\partial z}{\partial x}\right|_s \right) = 0$$

$$h\frac{\partial u}{\partial x} + \frac{\partial h\Omega}{\partial s} + \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + \frac{\partial u}{\partial s}\left(h\frac{\partial s}{\partial x} + \left.\frac{\partial z}{\partial x}\right|_s \right) = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial h\Omega}{\partial s} = 0$$

Coordonnées verticales généralisées : formulation

$$\left\{\begin{array}{rcl}\dfrac{\partial hu}{\partial t}+\dfrac{\partial hu^2}{\partial x}+\dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0}h\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right]\\ \dfrac{\partial h}{\partial t}+\dfrac{\partial hu}{\partial x}+\dfrac{\partial h\Omega}{\partial s} & = & 0\\ \dfrac{\partial p}{\partial s} & = & -\rho g h\\ \dfrac{\partial h\rho}{\partial t}+\dfrac{\partial hu\rho}{\partial x}+\dfrac{\partial h\Omega\rho}{\partial s} & = & 0\end{array}\right.$$

$$h=\frac{\partial z}{\partial s}$$

$$\left\{\begin{array}{rcl}\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+w\dfrac{\partial u}{\partial z} & = & -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x}\\ \dfrac{\partial u}{\partial x}+\dfrac{\partial w}{\partial z} & = & 0\\ \dfrac{\partial p}{\partial z} & = & -\rho g\\ \dfrac{\partial \rho}{\partial t}+u\dfrac{\partial \rho}{\partial x}+w\dfrac{\partial \rho}{\partial z} & = & 0\end{array}\right.$$

Plan

1. Coordonnées généralisées : formulation des équations

2. **Choix des coordonnées verticales**

3. Exemples de coordonnées verticales

Coordonnées verticales généralisées : formulation

$$\left\{\begin{array}{rcl}
\dfrac{\partial hu}{\partial t}+\dfrac{\partial hu^2}{\partial x}+\dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0}h\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\
\dfrac{\partial h}{\partial t}+\dfrac{\partial hu}{\partial x}+\dfrac{\partial h\Omega}{\partial s} & = & 0 \\
\dfrac{\partial p}{\partial s} & = & -\rho g h \\
\dfrac{\partial h\rho}{\partial t}+\dfrac{\partial hu\rho}{\partial x}+\dfrac{\partial h\Omega\rho}{\partial s} & = & 0
\end{array}\right.$$

On spécifie la coordonnée verticale h

On déduit la vitesse verticale

$$h\Omega = -\int_{s_{\text{bottom}}}^{s}\left[\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}\right]$$

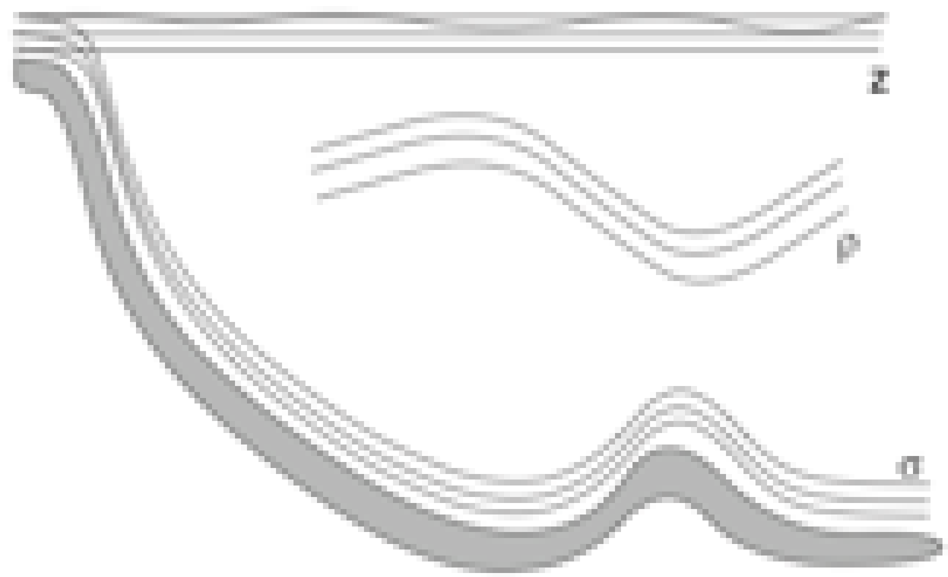
On spécifie la vitesse verticale Ω

$$\Omega = 0$$

On déduit l'évolution de la coordonnée verticale

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0$$

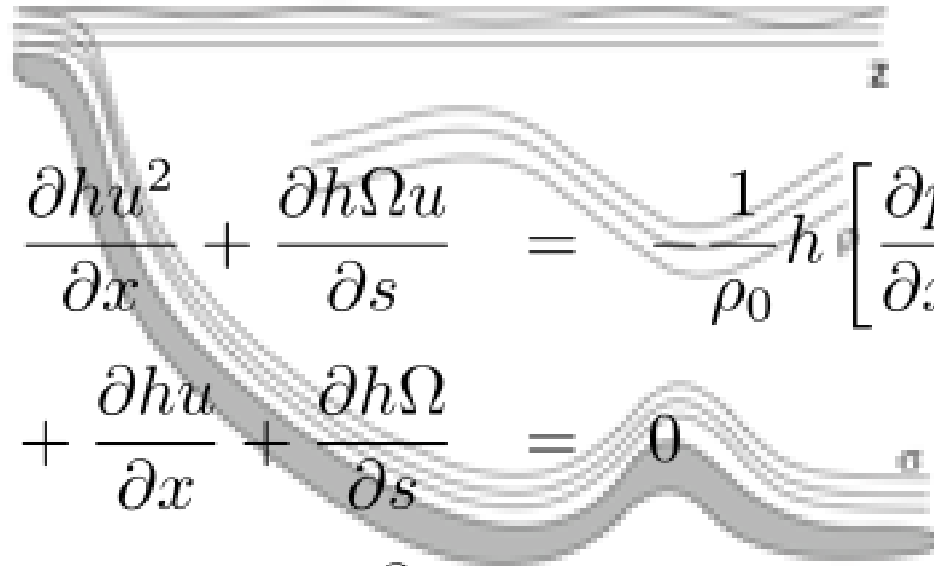
Différents types de coordonnées verticales



Quelques questions (I) :

- Est-ce que la bathymétrie est bien représentée ? (couche limite de fond)
- Est-ce que la couche de mélange est bien représentée ?
- Est-ce que dans l'océan intérieur, il n'y a pas trop de mélange diapycnal purement numérique ? (lié au non alignement des couches avec les isopycnes)

Différents types de coordonnées verticales



$$\left\{\begin{array}{rcl} \dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0} h \left[\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} + \dfrac{\partial h\Omega}{\partial s} & = & 0 \\ \dfrac{\partial p}{\partial s} & = & -\rho g h \\ \dfrac{\partial h\rho}{\partial t} + \dfrac{\partial hu\rho}{\partial x} + \dfrac{\partial h\Omega\rho}{\partial s} & = & 0 \end{array}\right.$$

$$\left.\frac{\partial p}{\partial x}\right|_z \longrightarrow \left.\frac{\partial p}{\partial x}\right|_s + \frac{\partial p}{\partial s}\frac{\partial s}{\partial x}$$

Quelques questions (II) :

- Est-ce que le gradient de pression <u>horizontal</u> est bien estimé ? (Est-ce que une stratification neutre reste au repos ?)
- Dans le cas non hydrostatique (solveur de Poisson pour la pression), est-ce que la matrice est bien conditionnée ?

Plan

1. Coordonnées généralisées : formulation des équations

2. Choix des coordonnées verticales

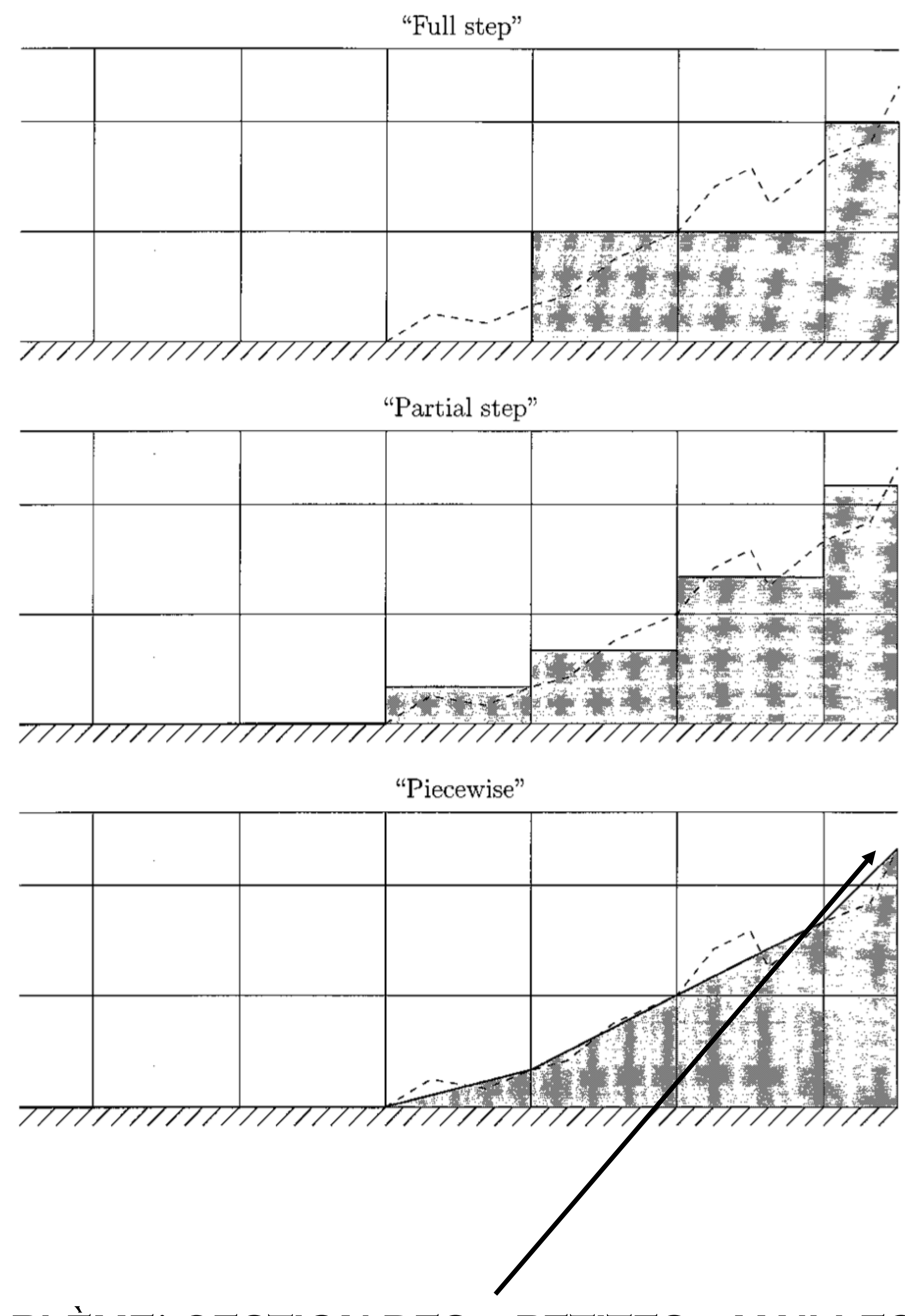
3. **Exemples de coordonnées verticales**

Coordonnées géopotentielles

- Avantages :
  - Calcul du gradient (horizontal) de pression (un seul terme)
- Inconvénients :
  - Diffusion diapycnale
  - Représentation du fond
  - Points masqués (perte d'efficacité)

Améliorations :

- Partial cells
- Shaved cells (Adcroft et al, 1997, MWR)



Problème: gestion des « petites » mailles

Coordonnées géopotentielles

Problème: gestion des « petites » mailles : regroupement

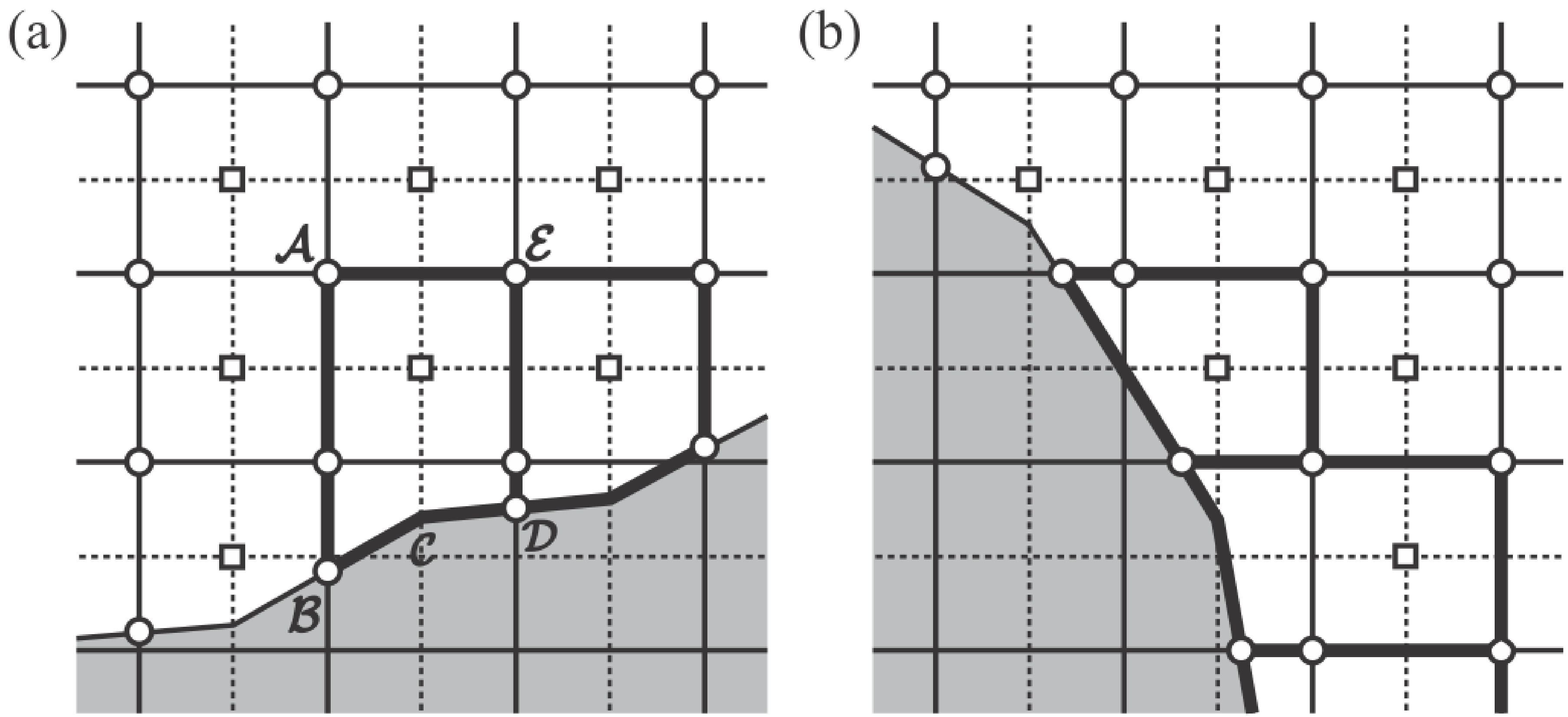
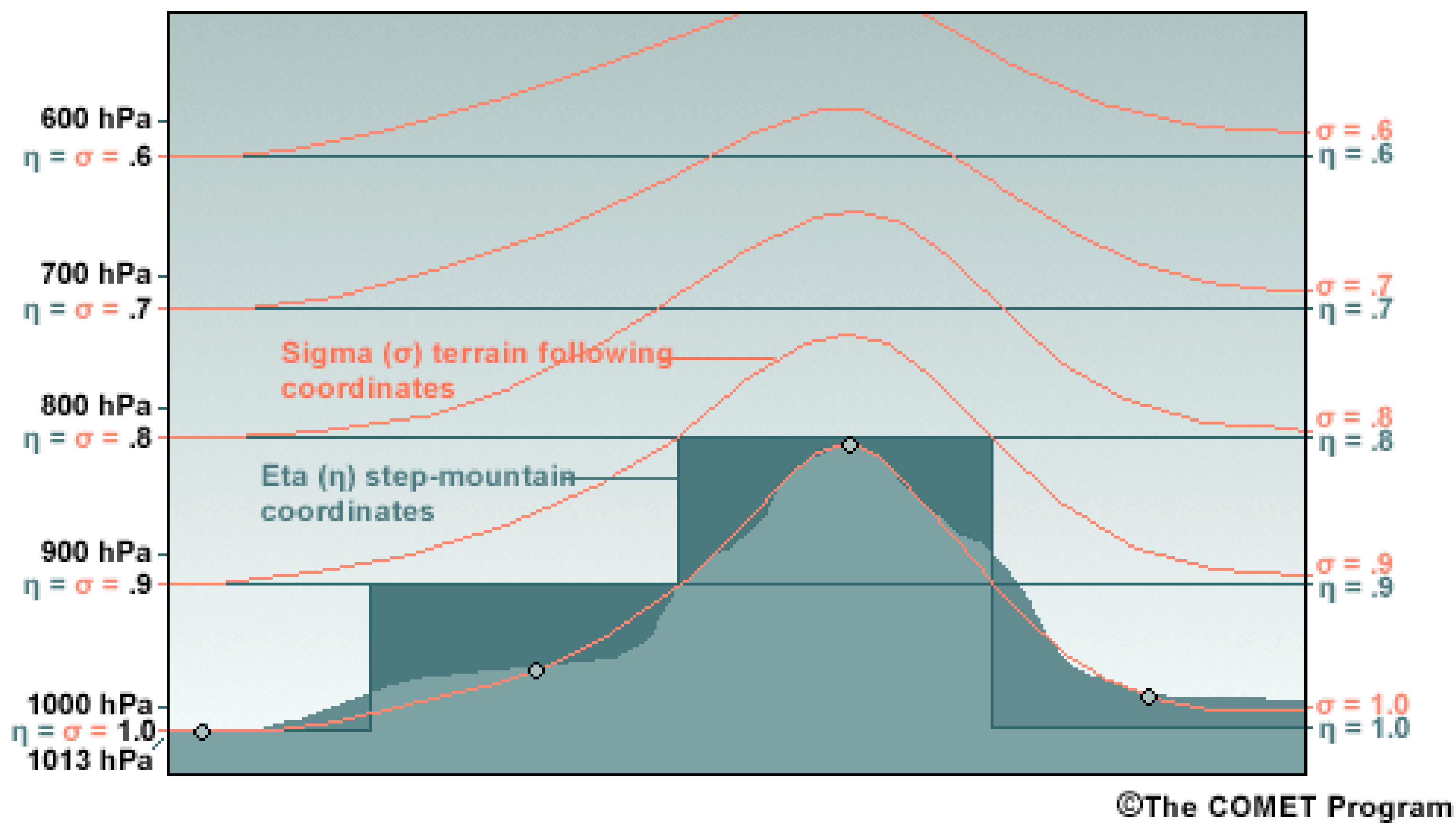


Fig. 2. Cell combination and variable arrangement. Thick lines describe the boundaries of combined cells. Squares and circles represent scalar points and velocity points, respectively. Shaded region represents the topography in the model. (a) Scalar cells on gentle slopes. (b) Scalar cells on steep slopes.

Yamazaki H. and T. Satomura, 2010, MWR.

Coordonnées sigma

- Avantages :
  - Condition à la limite au fond
  - Tous les points de calcul sont en mer
- Inconvénients :
  - Calcul du gradient de pression
  - Diffusion diapycnale
  - Système elliptique non hydrostatique



Calcul du gradient de pression

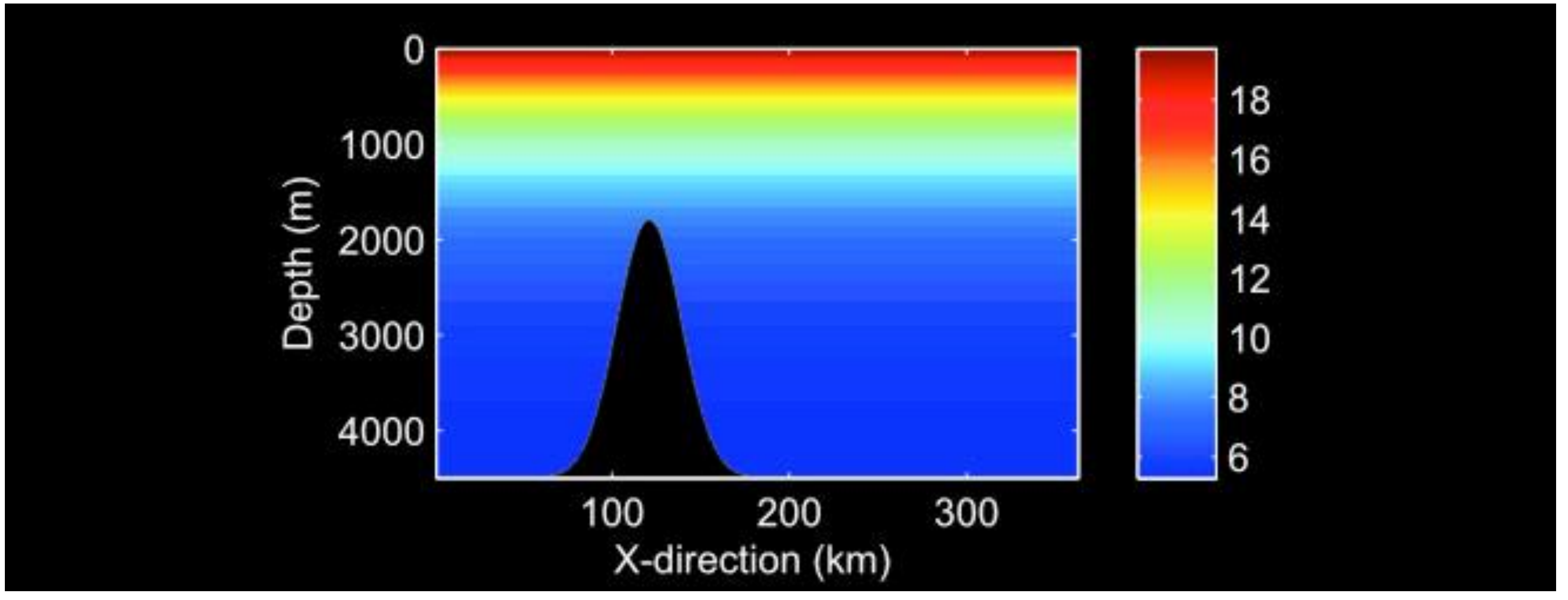
$$\left\{\begin{array}{rcl}
\dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0} h \left[\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\
\dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} + \dfrac{\partial h\Omega}{\partial s} & = & 0 \\
\dfrac{\partial p}{\partial s} & = & -\rho g h \\
\dfrac{\partial h\rho}{\partial t} + \dfrac{\partial hu\rho}{\partial x} + \dfrac{\partial h\Omega\rho}{\partial s} & = & 0
\end{array}\right.$$

2 termes

Calcul du gradient de pression

Stratification neutre - coordonnées sigma

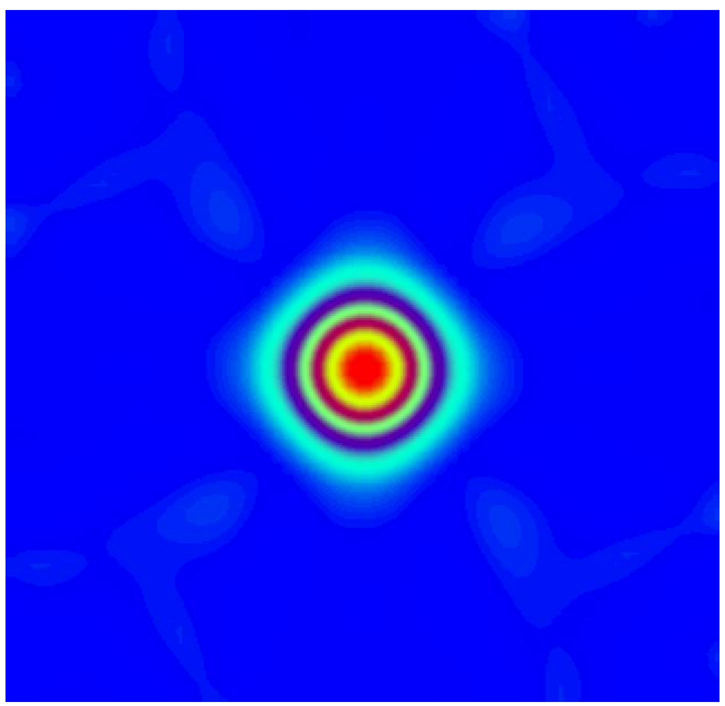
Test du mont sous-marin



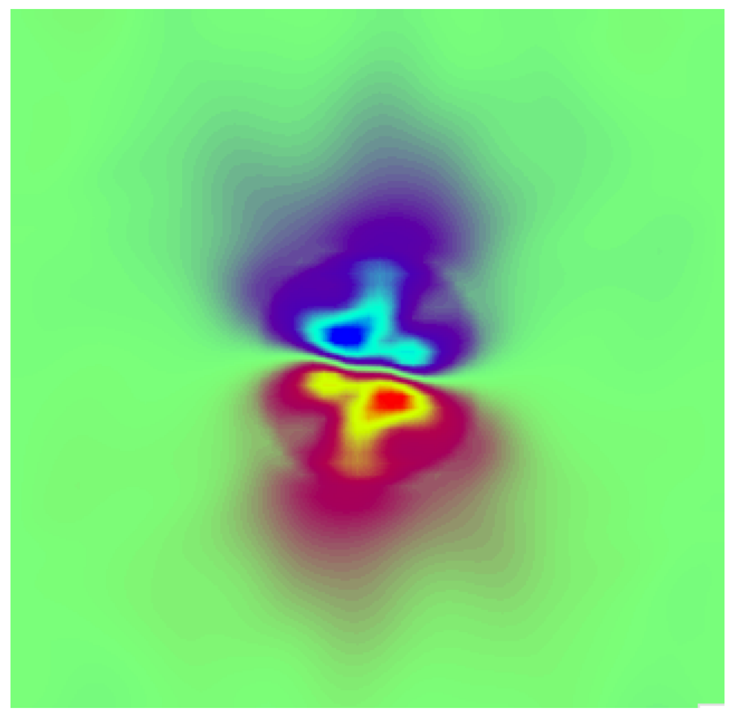
Calcul du gradient de pression

Stratification neutre - coordonnées sigma

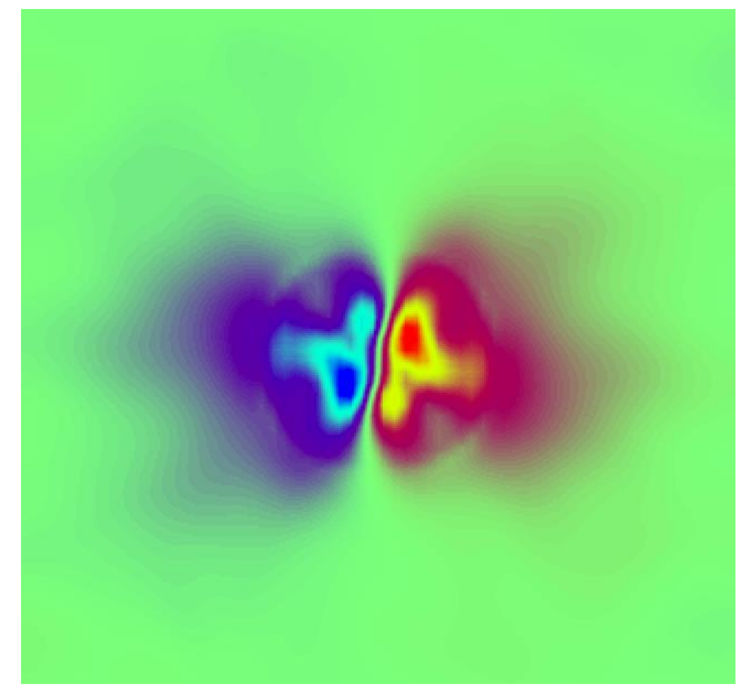
Test du mont sous-marin



ζ



U



V

élévation de surface

Courants barotropes

Calcul du gradient de pression

$$
\begin{cases}
\dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0} h \left[\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x} \right] \\
\dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} + \dfrac{\partial h\Omega}{\partial s} & = & 0 \\
\dfrac{\partial p}{\partial s} & = & -\rho g h \\
\dfrac{\partial h\rho}{\partial t} + \dfrac{\partial hu\rho}{\partial x} + \dfrac{\partial h\Omega\rho}{\partial s} & = & 0
\end{cases}
$$

2 termes

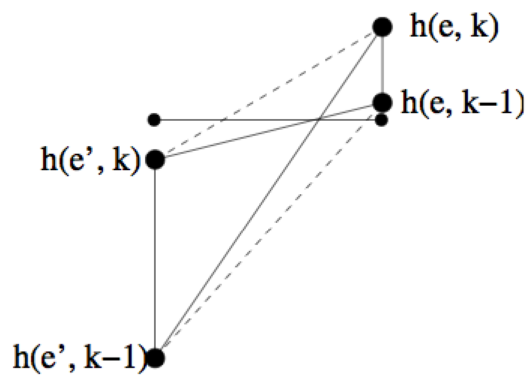
- On retire un $\bar{\rho}(z)$
- On utilise des schémas d'ordre élevés
- On modifie l'équation d'état
- On lisse la bathymétrie

Shchepetkin A. and J.C. Mc Williams, 2011: Accurate Boussinesq oceanic modeling with a practical, "Stiffened" Equation of State, Ocean Modelling

Lissage de la bathymétrie

$$r^0(i,j) = \frac{|H_{i+1} - H_i|}{|H_{i+1} + H_i|} \le r^0_{\max} \le 0.2$$

- Denote by $C_k(e)$ the parallelepiped of water between depth $h(e, k-1)$ and depth $h(e, k)$.
- Hydrostatic consistency means that if e and e' are any two adjacent cells, then $C_k(e)$ and $C_k(e')$ share a level.



$$rx_1(h, e, e', k) = \frac{|h(e,k) - h(e',k) + h(e,k-1) - h(e',k-1)|}{h(e,k) + h(e',k) - h(e,k-1) - h(e',k-1)}.$$

- To impose that $C_k(e)$ and $C_k(e')$ share a level is equivalent to $rx_1(h, e, e', k) \le 1$ (Rousseau and Pham 1971, Mesinger 1982, Haney 1991).
- This requirement is very strong and almost impossible to fulfill.

$$rx_1 \le 3$$

From Matthieu Dutour

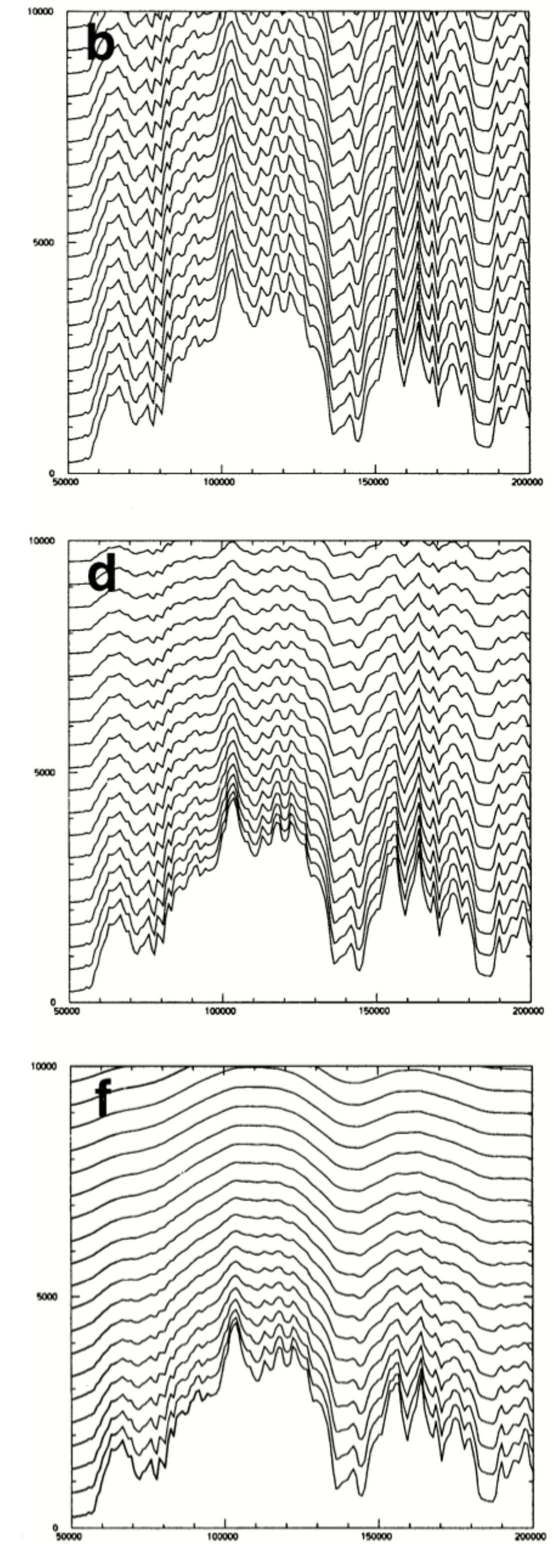
Coordonnées sigma

Amélioration :

Séparation topo lisse, topo non lisse

Schär et al, 2002 : A new Terrain-Following Vertical Coordinate Formulation for Atmospheric Prediction Models, MWR

h=h_smooth + (h-h_smooth)

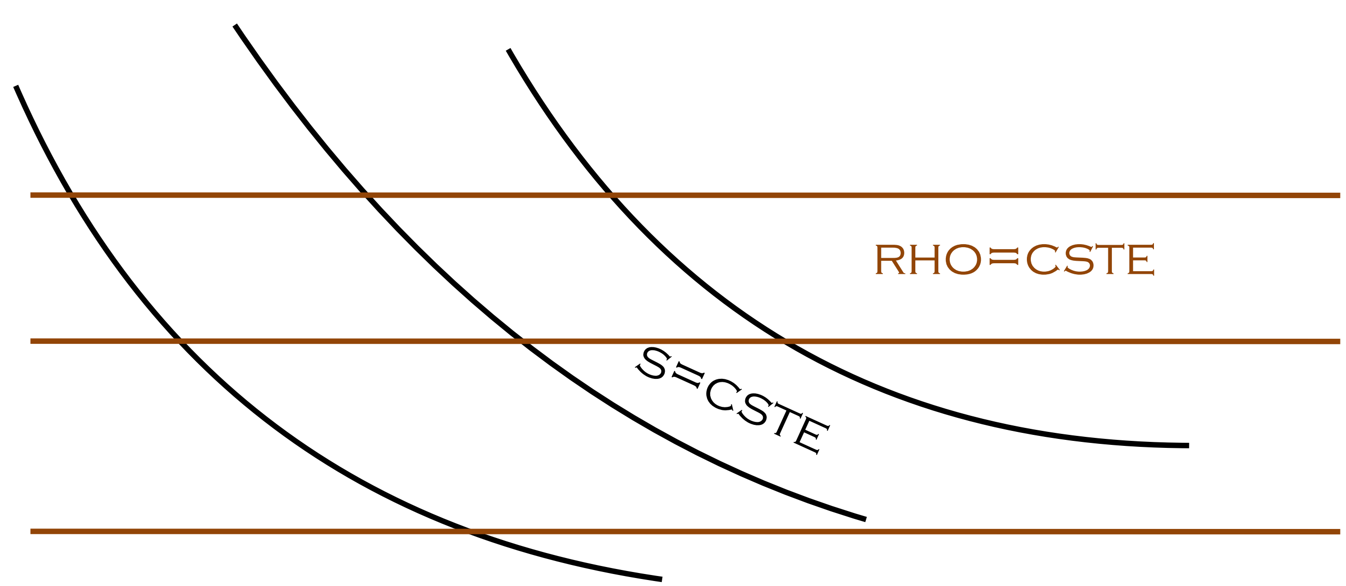


Diffusion diapycnale artificielle

Utilisation de schémas « horizontaux » d'advection (de traceurs) diffusifs (décentrés)

↓

La diffusion a lieu le long des couches du modèle

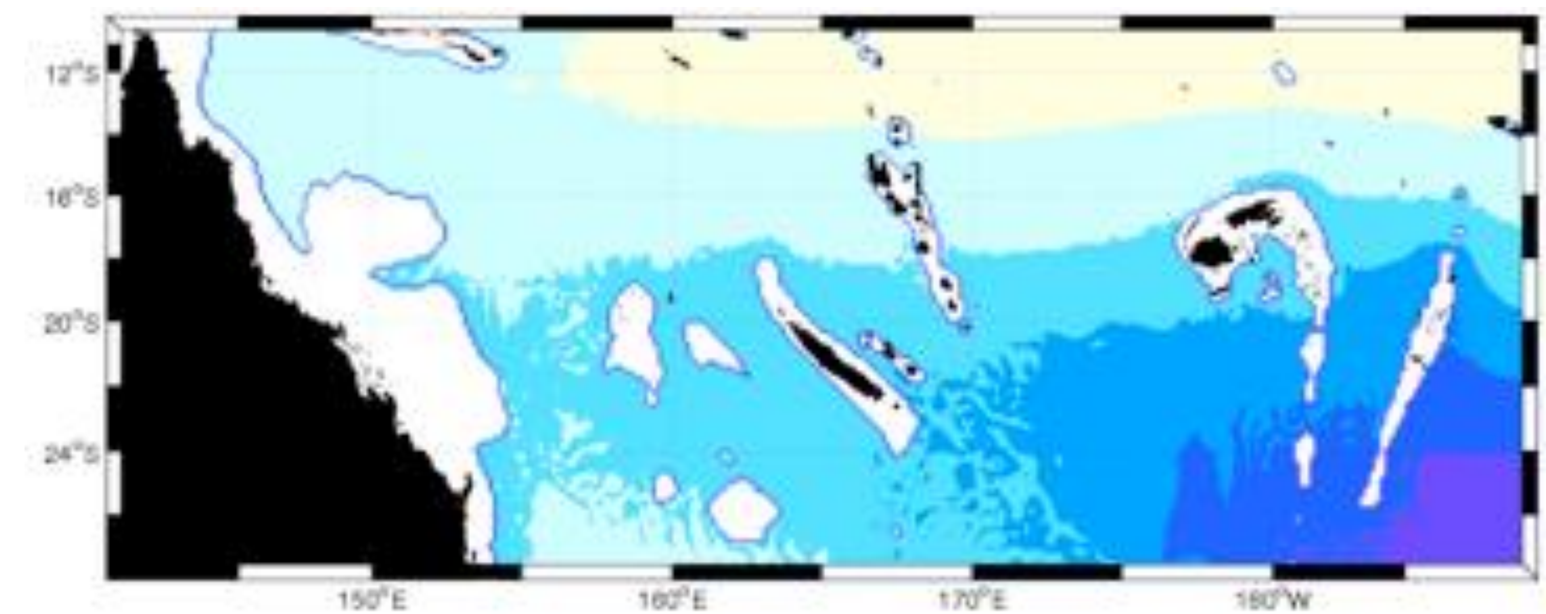


Diffusion diapycnale artificielle

Marchesiello et al, 2009, Ocean Modelling

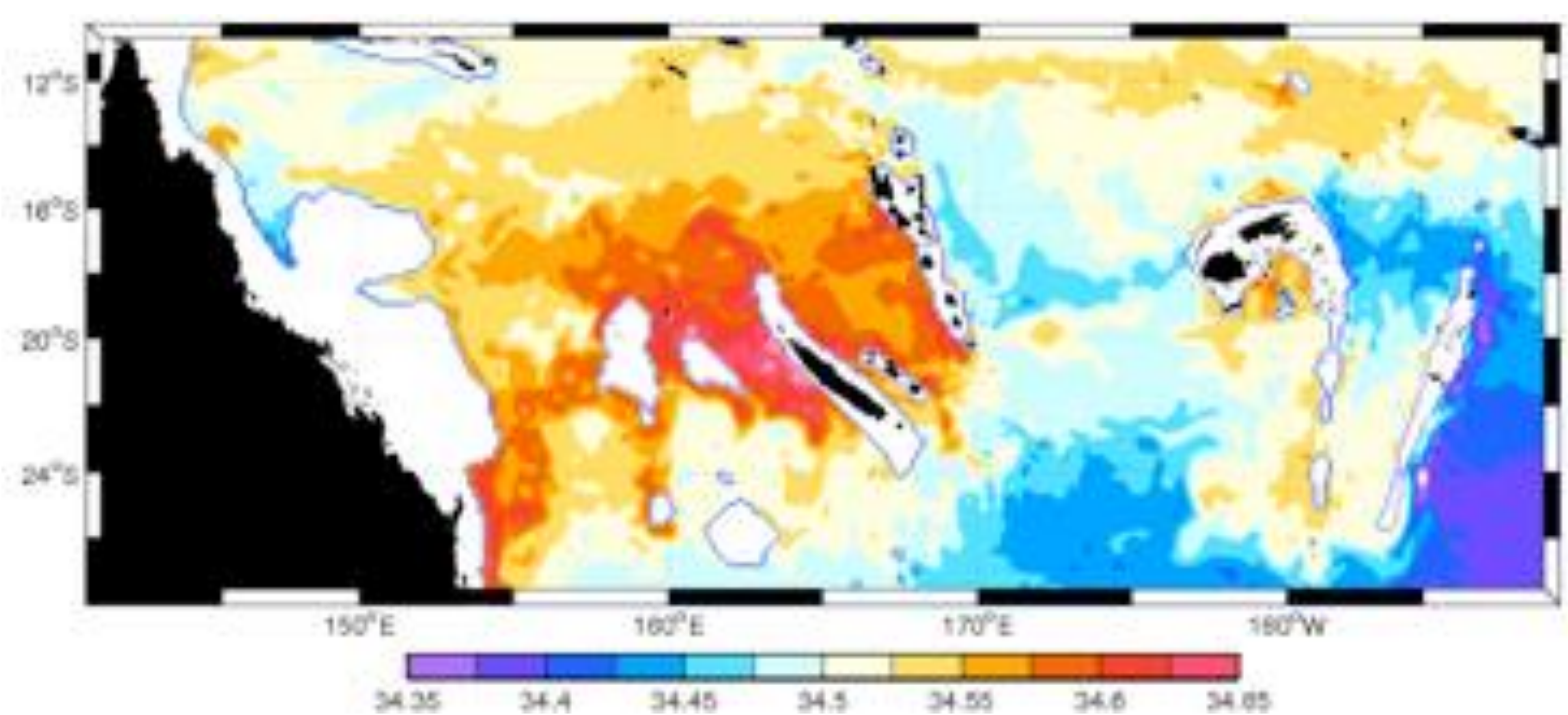
Salinité à 1000m

Climatologie



ROMS

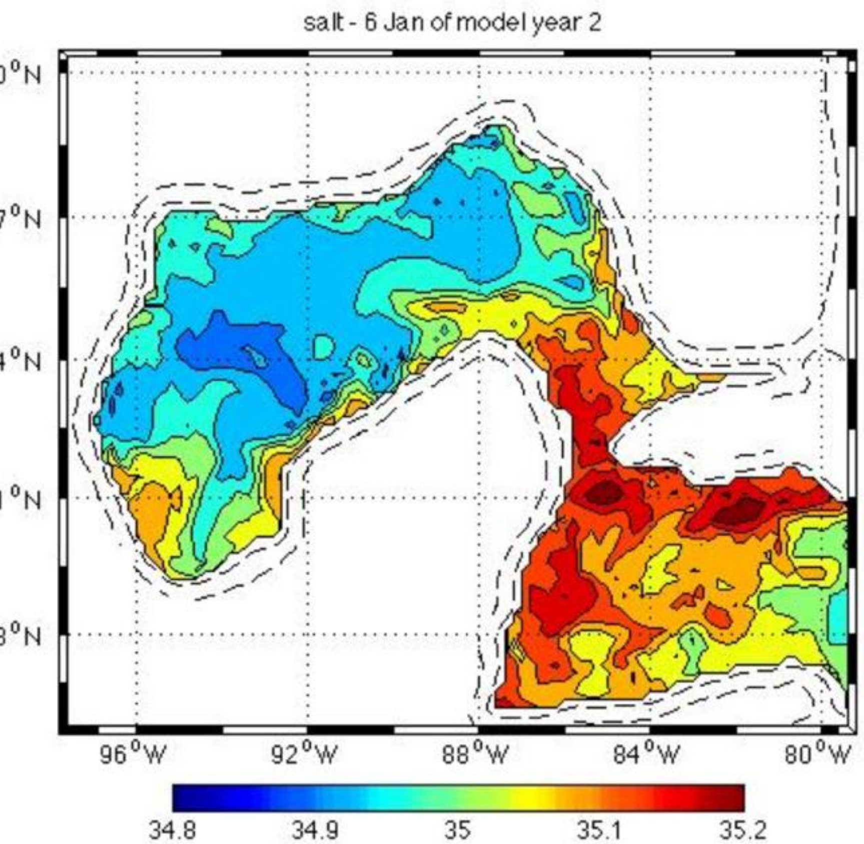
Advection : décentré 3ème ordre



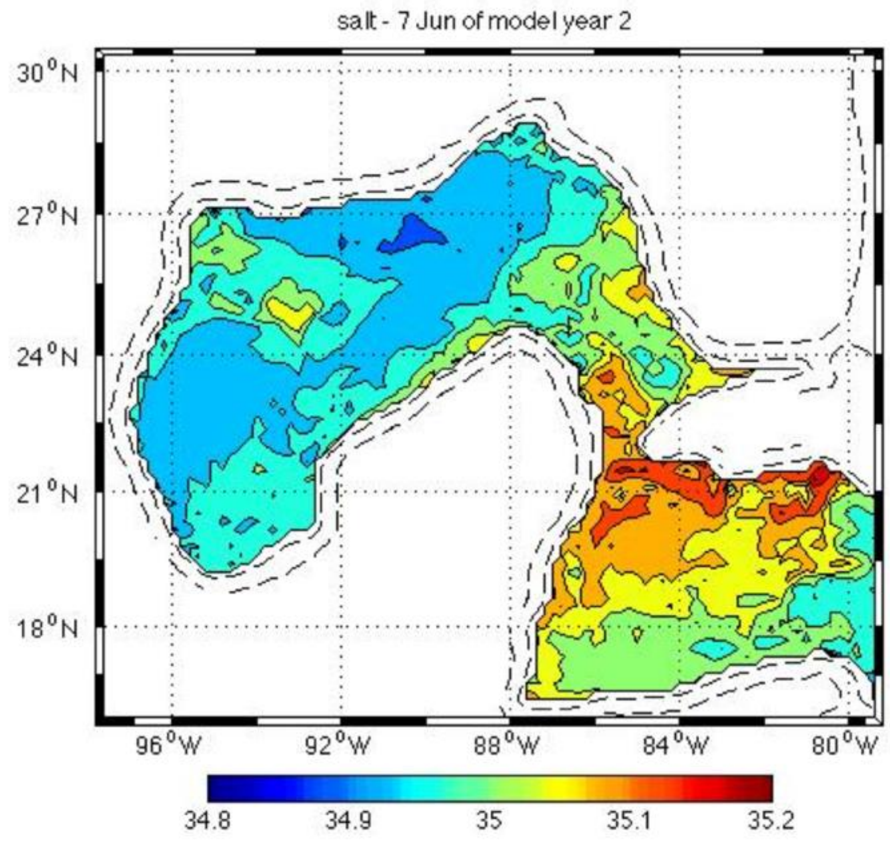
Advection et diffusion implicite

Salinité à 1000m

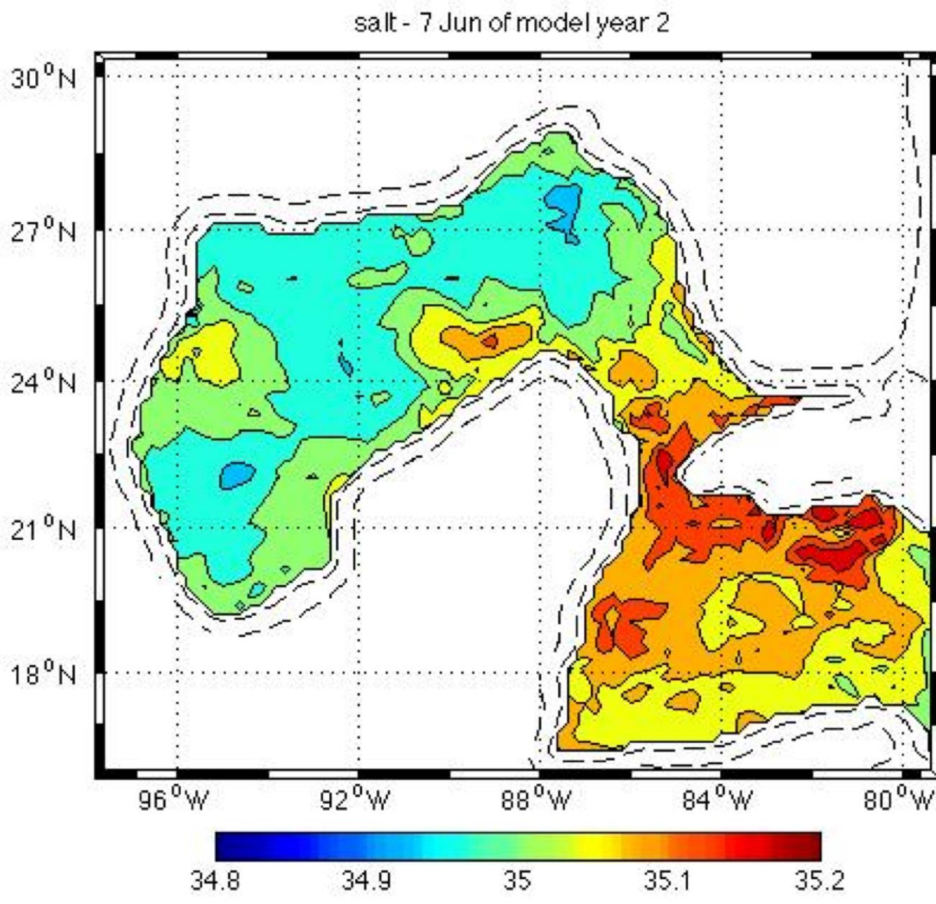
ROMS UP3



ROMS UP5



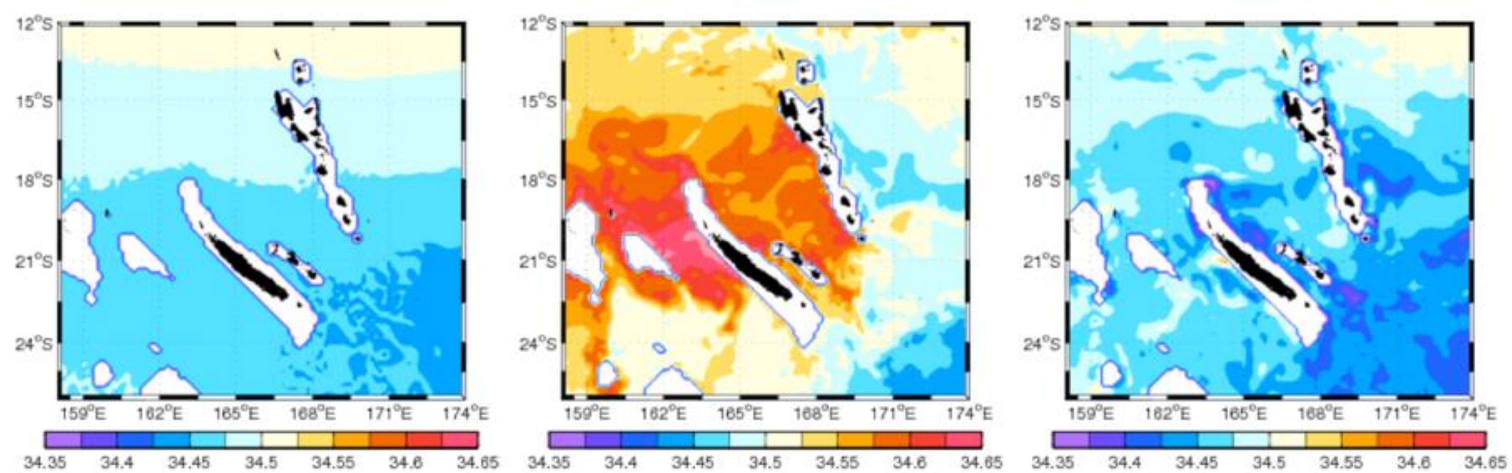
ROMS WENO5

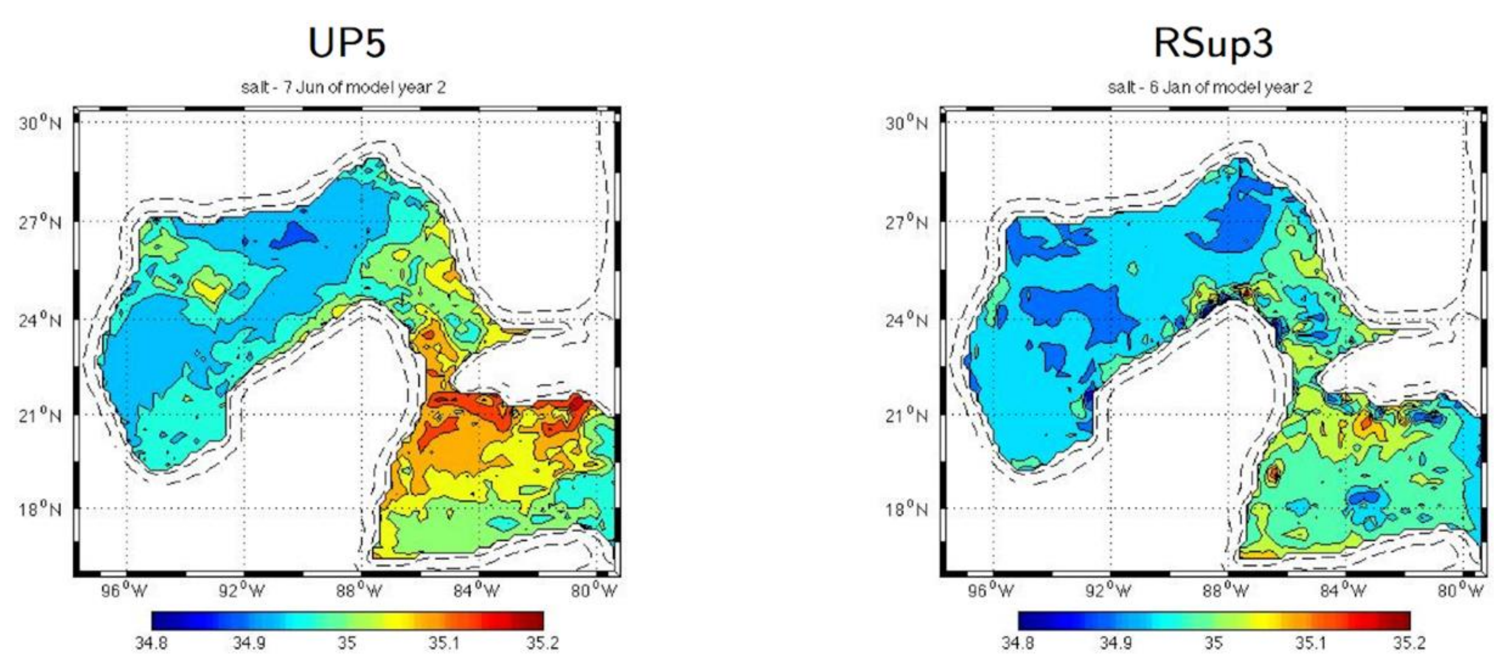


Advection et diffusion implicite

Rotate the diffusive part of the upstream biased scheme :

$$\left.\frac{\partial q}{\partial x}\right|_{(2n+1)order} = \left.\frac{\partial q}{\partial x}\right|_{(2n+2)order} + (-1)^n K \underbrace{\frac{\partial^{(2n+2)} q}{\partial x^{2n+2}}}_{rotated}$$





Advection et diffusion implicite

Rotate the diffusive part of the upstream biased scheme :

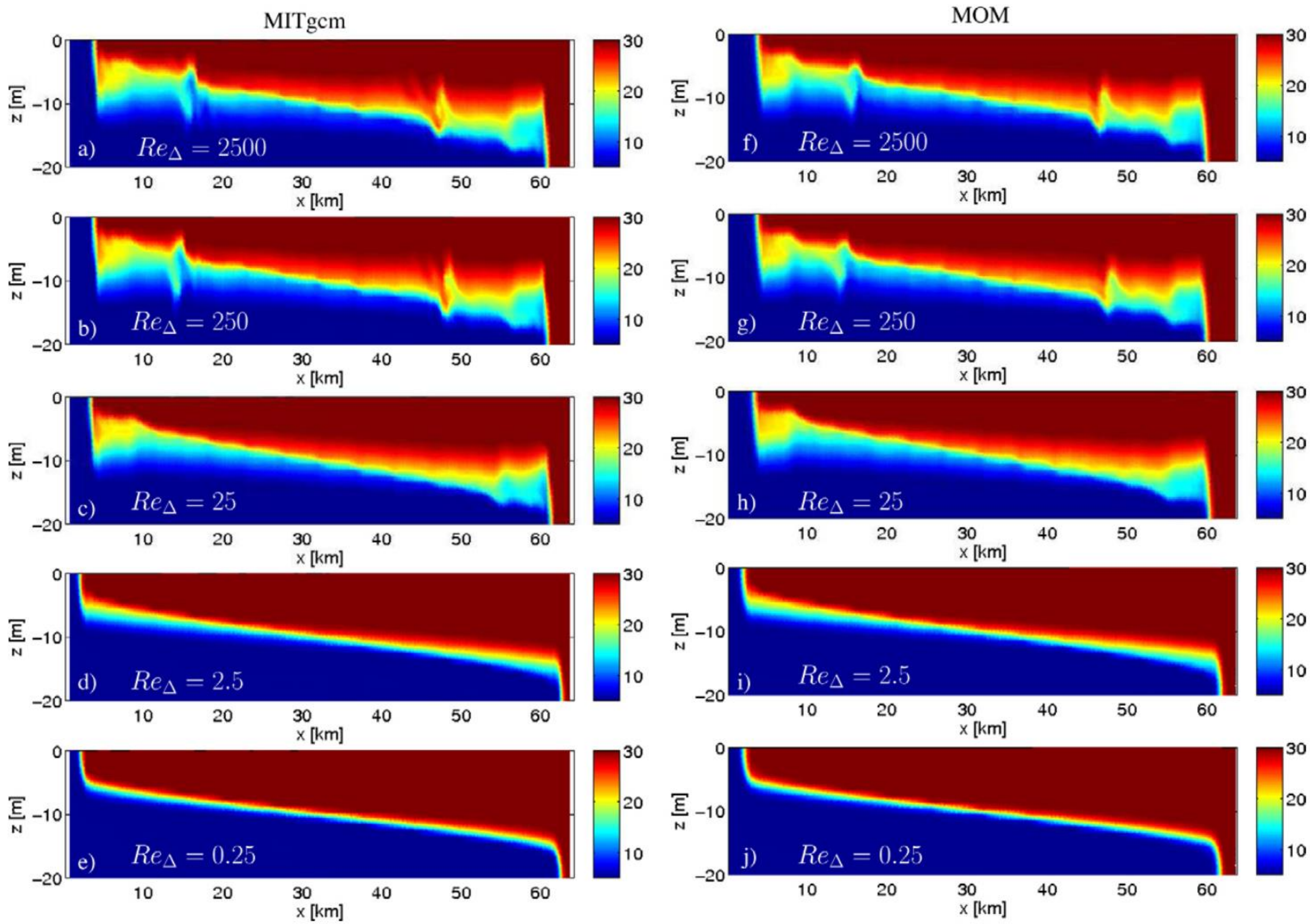
$$\left.\frac{\partial q}{\partial x}\right|_{(2n+1)order} = \left.\frac{\partial q}{\partial x}\right|_{(2n+2)order} + (-1)^n K \underbrace{\frac{\partial^{(2n+2)} q}{\partial x^{2n+2}}}_{rotated}$$

Stabilisation de ce terme (qui inclue de la diffusion verticale) ?

Marchesiello et al, 2009 : Spurious diapycnal mixing: The problem and a solution, *Ocean Modelling*

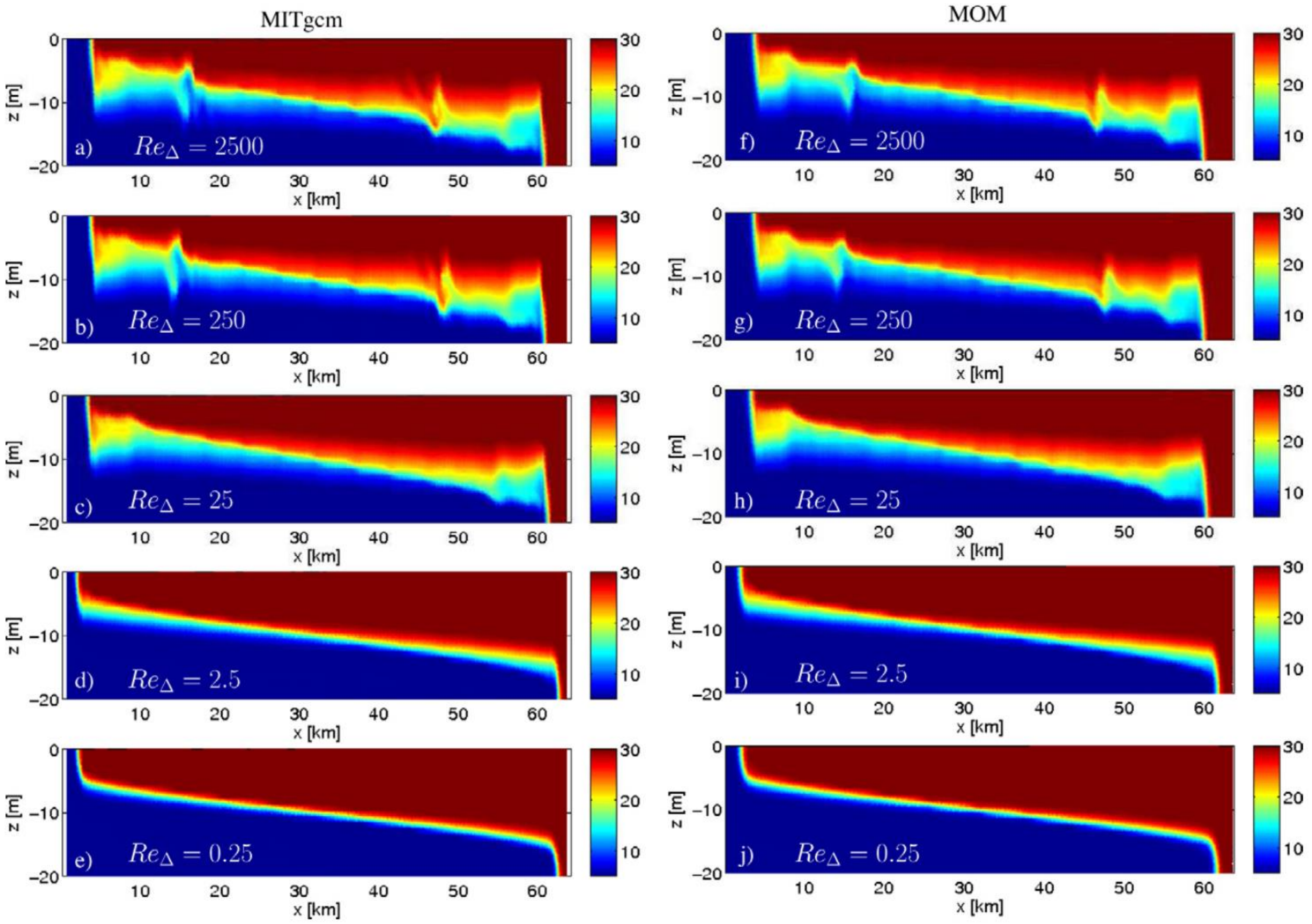
Lemarié et al, 2012: On the stability and accuracy of the harmonic and biharmonic isoneutral mixing operators in ocean models. *Ocean Modelling*

Impact de la viscosité



Illicak et al, 2011: Spurious dianeutral mixing and the role of momentum closure, Ocean Modeling

Impact de la viscosité



Illicak et al, 2011: Spurious dianeutral mixing and the role of momentum closure, Ocean Modelling

→ Faut il utiliser des schémas monotones sur le moment ?

Coordonnées isopycnales / Coordonnées hybrides

$$\left\{\begin{array}{rcl} \dfrac{\partial hu}{\partial t}+\dfrac{\partial hu^2}{\partial x}+\dfrac{\partial h\Omega u}{\partial s} & = & -\dfrac{1}{\rho_0}h\left[\dfrac{\partial p}{\partial x}+\dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x}\right] \\ \dfrac{\partial h}{\partial t}+\dfrac{\partial hu}{\partial x}+\dfrac{\partial h\Omega}{\partial s} & = & 0 \\ \dfrac{\partial p}{\partial s} & = & -\rho g h \\ \dfrac{\partial h\rho}{\partial t}+\dfrac{\partial hu\rho}{\partial x}+\dfrac{\partial h\Omega\rho}{\partial s} & = & 0 \end{array}\right.$$

$$s=\rho$$

$$\Omega=\left.\frac{\partial s}{\partial t}\right|_z+u\left.\frac{\partial s}{\partial x}\right|_z+w\frac{\partial s}{\partial z}=0$$

$P=p+\rho g z$ Potentiel de Montgomery

$$\begin{array}{rcl} \dfrac{\partial u}{\partial t}+u\left.\dfrac{\partial u}{\partial x}\right|_s & = & -\dfrac{1}{\rho_0}\left.\dfrac{\partial P}{\partial x}\right|_s \\ \dfrac{\partial h}{\partial t}+\left.\dfrac{\partial hu}{\partial x}\right|_s & = & 0 \\ \dfrac{\partial P}{\partial s} & = & -gz \\ \dfrac{\partial \rho}{\partial t}+u\left.\dfrac{\partial \rho}{\partial x}\right|_s & = & 0 \end{array}$$

- Avantages :
  - Diffusion diapycnale
- Inconvénients :
  - Résolution verticale dans les zones peu stratifiées

Coordonnées isopycnales / Coordonnées hybrides

$P = p + \rho g z$ Potentiel de Montgomery

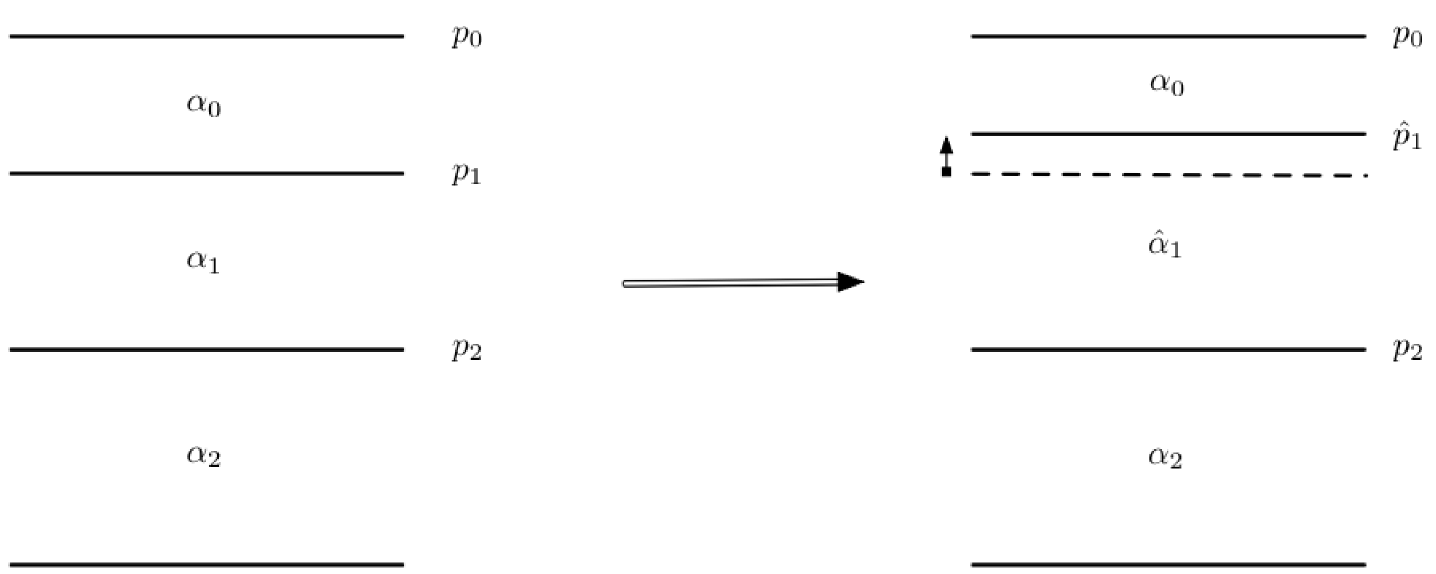
$$\begin{aligned}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\bigg|_s &= -\frac{1}{\rho_0} \frac{\partial P}{\partial x}\bigg|_s \\
\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x}\bigg|_s &= 0 \\
\frac{\partial P}{\partial s} &= -gz \\
\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x}\bigg|_s &= 0
\end{aligned}$$

← En pratique, non nul (forçages)

→ On contraint les couches verticales à rester isopycnales

Coordonnées isopycnales / Coordonnées hybrides

Le premier niveau (1) est trop dense (par rapport à sa densité cible)
-> On déplace son interface supérieure vers le haut (apport d'eau légère dans la couche 1)



REZONING

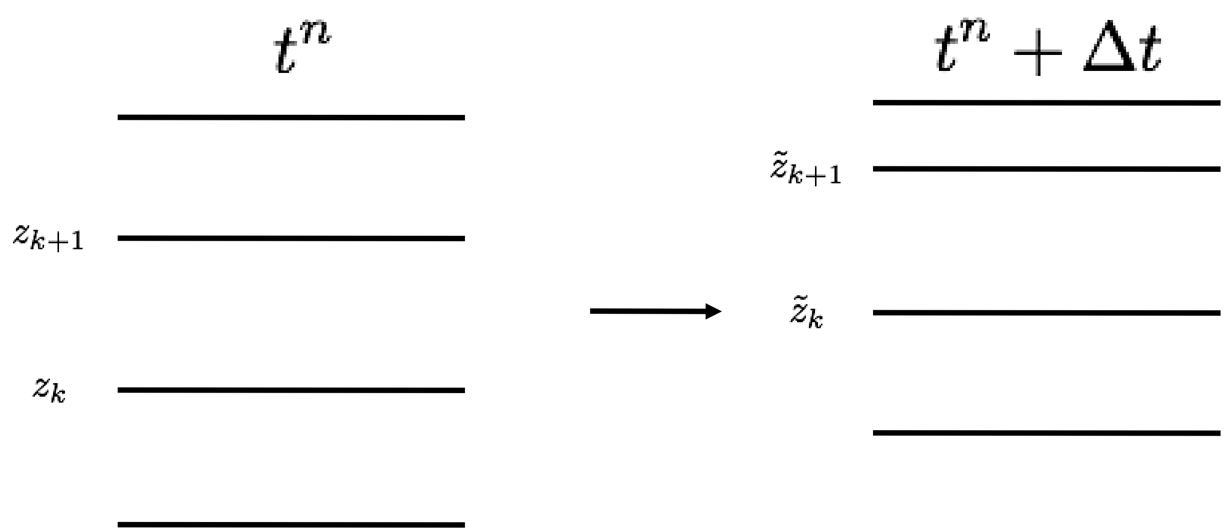
On obtient une nouvelle distribution des couches

puis REMAPPING vertical vers les nouvelles couches (i.e. schéma lagrangien)

$$\int_{V_a^{n+1}} T = \int_{V_d^n} T$$

Coordonnées isopycnales / Coordonnées hybrides

Remapping (conservatif)



On cherche $T(z)$ tel que

$$\frac{1}{z_{k+1} - z_k} \int_{z_k}^{z_{k+1}} T(z')\mathrm{d}z' = T_k$$

On déduit

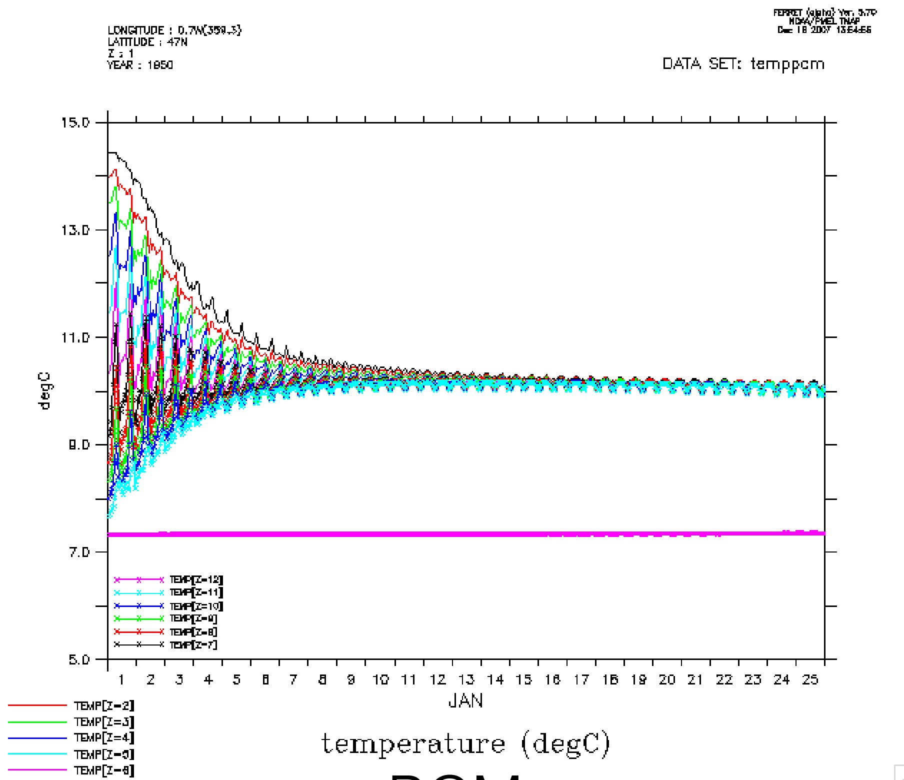
$$\tilde{T}_k = \frac{1}{\tilde{z}_{k+1} - \tilde{z}_k} \int_{\tilde{z}_k}^{\tilde{z}_{k+1}} T(z')\mathrm{d}z'$$

Par construction

$$\sum_k (\tilde{z}_{k+1} - \tilde{z}_k)\tilde{T}_k = \int_{z_{\mathrm{bottom}}}^{z_0} T(z')\mathrm{d}z' = \sum_k (z_{k+1} - z_k)T_k$$

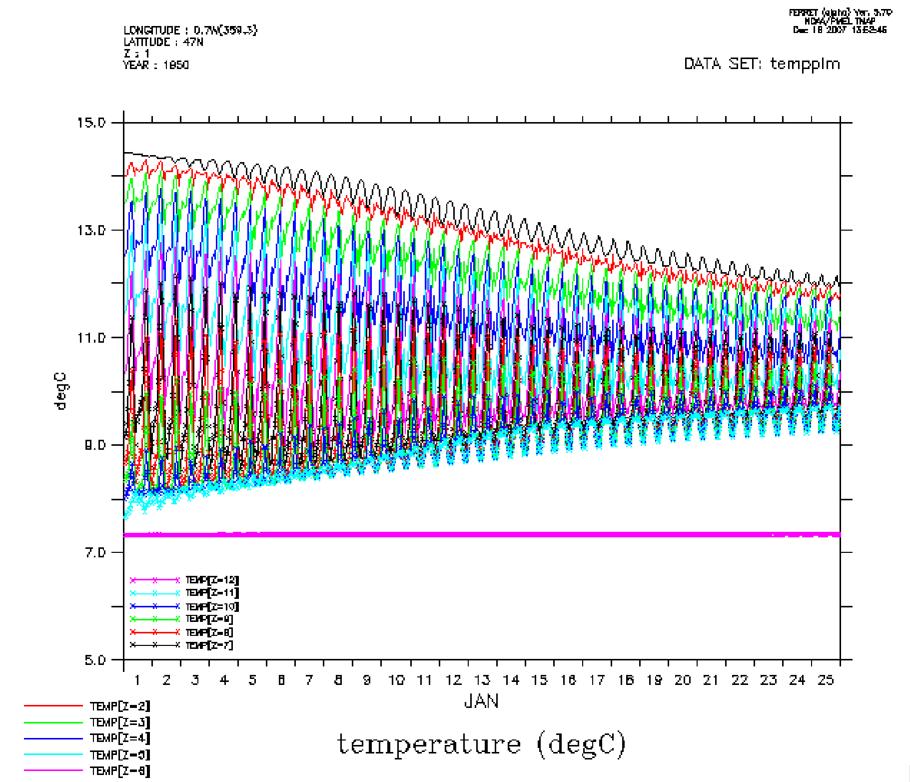
$\longrightarrow$ Conservation

Remapping

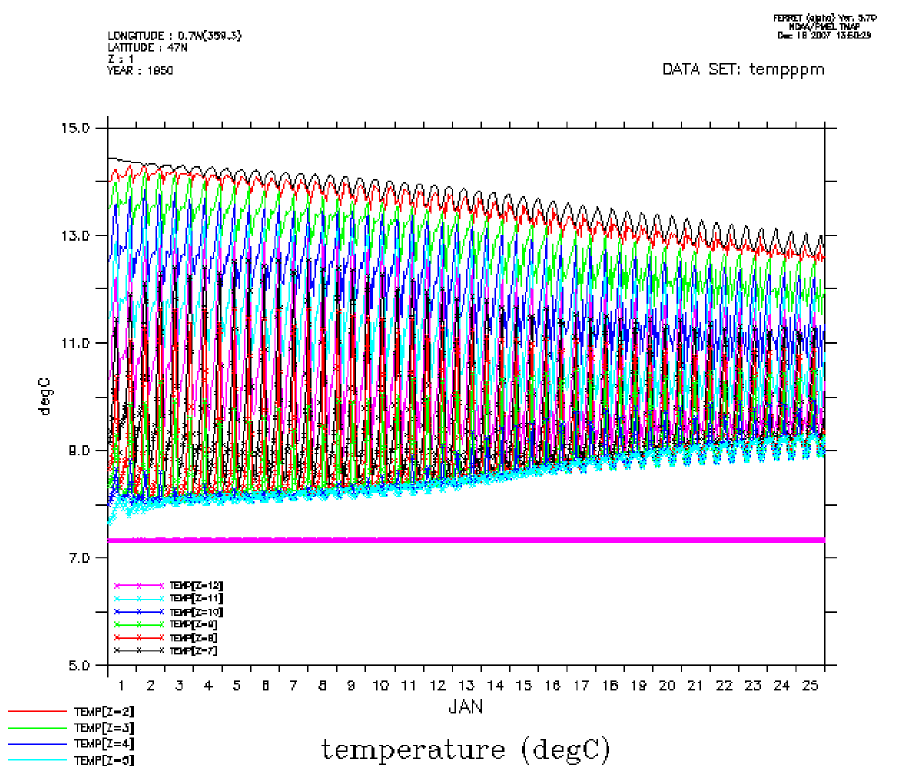


PCM

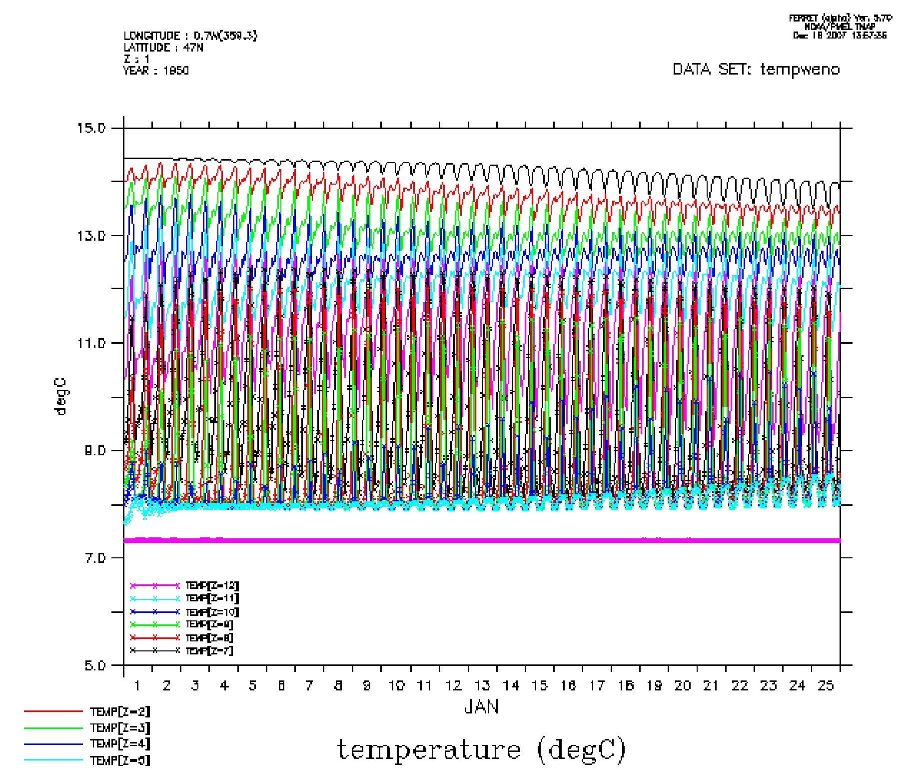
HYCOM



PLM



PPM



WENO

Coordonnées ztilde (NEMO, MPAS)

$$\begin{cases} \dfrac{\partial hu}{\partial t} + \dfrac{\partial hu^2}{\partial x} + \dfrac{\partial h\Omega u}{\partial s} &= -\dfrac{1}{\rho_0} h \left[\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial s}\dfrac{\partial s}{\partial x} \right] \\ \dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} + \dfrac{\partial h\Omega}{\partial s} &= 0 \\ \dfrac{\partial p}{\partial s} &= -\rho g h \\ \dfrac{\partial h\rho}{\partial t} + \dfrac{\partial hu\rho}{\partial x} + \dfrac{\partial h\Omega\rho}{\partial s} &= 0 \end{cases}$$

Objectif : éliminer les hautes fréquences de la vitesse verticale Ω

$$\frac{\partial h}{\partial t} = -\tilde{D}, \quad D = \frac{\partial hu}{\partial x}$$

$\tilde{D}$ composante haute fréquence de la divergence horizontale D

Leclair et Madec, 2011, Ocean Modelling