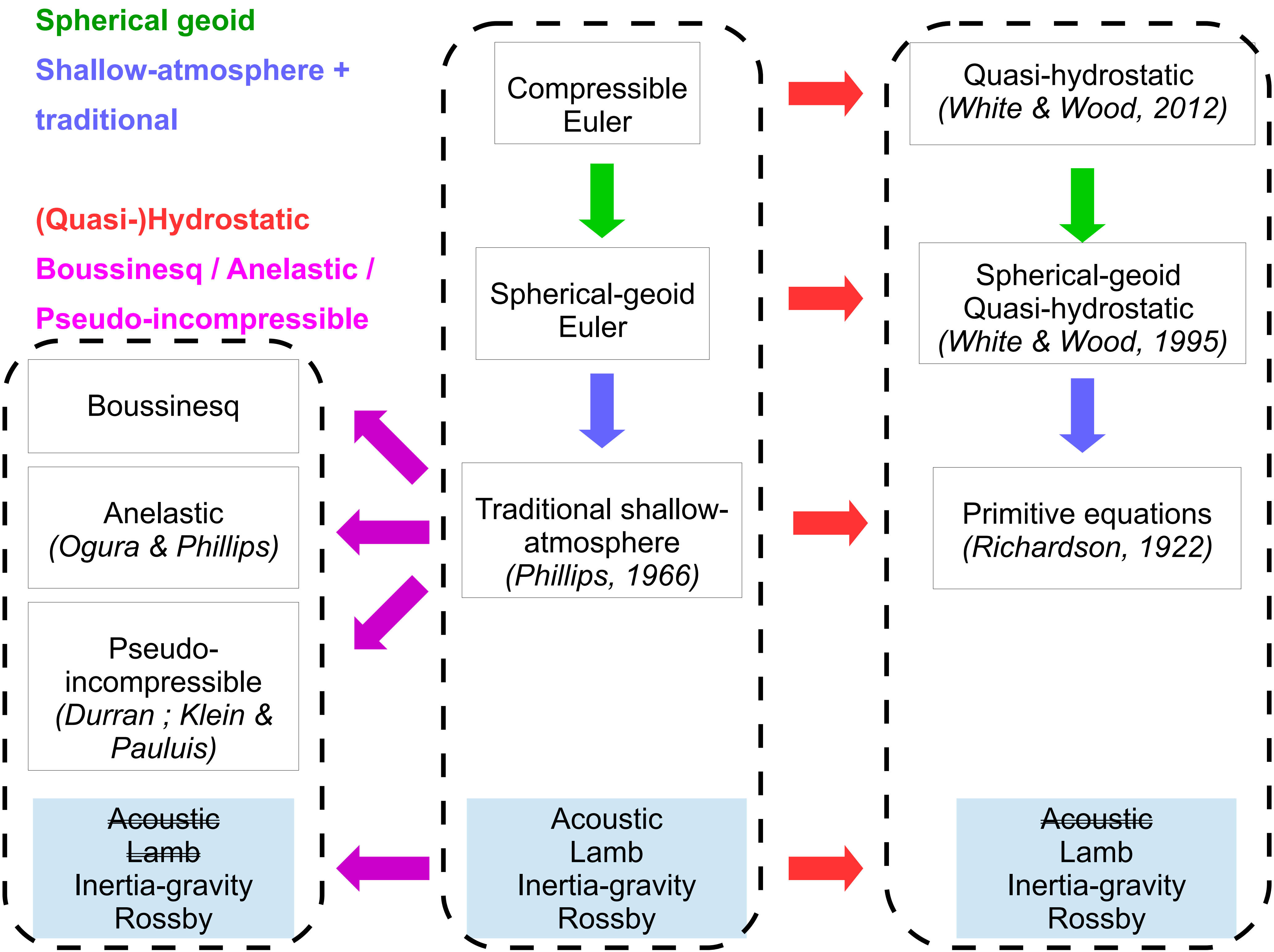


Spherical geoid
Shallow-atmosphere + traditional
(Quasi-)Hydrostatic
Boussinesq / Anelastic / Pseudo-incompressible
Compressible Euler
Spherical-geoid Euler
Traditional shallow-atmosphere (Phillips, 1966)
Quasi-hydrostatic (White & Wood, 2012)
Spherical-geoid Quasi-hydrostatic (White & Wood, 1995)
Primitive equations (Richardson, 1922)
Boussinesq
Anelastic (Ogura & Phillips)
Pseudo-incompressible (Durran ; Klein & Pauluis)
~~Acoustic~~
~~Lamb~~
Inertia-gravity
Rossby
Acoustic
Lamb
Inertia-gravity
Rossby
~~Acoustic~~
Lamb
Inertia-gravity
Rossby

Discretization I

What do you have to say about a numerical model/scheme ?

- Nice !
- Ugly !
- I like it !
- I hate it !

Hmm... let's try and be more factual

- Accurate
- Conservative
- Dispersive
- Dissipative / diffusive
- Positive / monotone
- Stable
- Explicit vs Implicit
- Local vs global

- We care about these properties because they are present in the continuous laws of motion
- Can we obtain them in a discretized model ? How ?
- All ? Independently ? Incompatibilities ?
- Implications for performance / parallelism ?

Context : Cartesian geometry

- Conservation laws for continuous models
- From the continuous to the discrete, and what you lose
- Numerical dispersion : illustration with the rotating shallow-water equations

Context : Cartesian geometry

- **Conservation laws for continuous models**
- From the continuous to the discrete, and what you lose
- Numerical dispersion : illustration with the rotating shallow-water equations

What they are

- Lagrangian vs integral
- Adiabatic vs diabatic
- Robust vs fragile

Where they come from

- Kinematics vs dynamics
- Variational principles, symmetries and dynamical conservation laws

What they imply

- stability

Conservation laws for continous models

Integral conservation law :

- Mass
- Total water / other species
- Energy
- (angular) momentum
- Potential enstrophy

$$\mathcal{M} = \int \rho \mathrm{d}x \mathrm{d}y \mathrm{dz} \qquad \frac{\mathrm{d}\mathcal{M}}{\mathrm{d}t} = \text{boundary terms}$$

$$\mathcal{H} = \int \left(\frac{u^2 + v^2 + w^2}{2} + gz + e\left(\frac{1}{\rho},\, s,\, r\right)\right) \rho \mathrm{d}x \mathrm{d}y \mathrm{dz}$$

$$\mathcal{L} = \int a^2 \cos^2 \Phi \left(\Omega + \dot{\lambda}\right) \mu \mathrm{d}\lambda \mathrm{d}\phi \mathrm{dz}$$

$$\mathcal{Z} = \int \rho q^2 \mathrm{d}x \mathrm{d}y \mathrm{dz}$$

Lagrangian conservation law : examples

- Entropy
- Water / other species
- Potential vorticity

$$\frac{Ds}{Dt} = 0 \qquad \frac{Dq}{Dt} = 0$$

Lagrangian conservation laws => infinitely many integral conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f(q,s) \rho \mathrm{d}x \mathrm{d}y \mathrm{dz} = \text{boundary terms}$$

Example : potential vorticity => potential enstrophy

Conservation laws : adiabatic vs diabatic

- irreversible processus => extra source / flux terms
  - total energy : heating
  - momentum : viscous stresses (boundary)
- Exception : mass

Actual physics are irreversible but very little : frictionless limit
Relevant conservation laws ?
cf Thuburn, JCP 2008

https://wiki.ucar.edu/download/attachments/25037023/14-Thuburn-Conservation.pdf?api=v2

Focus on adiabatic conservation laws, i.e. those that hold when irreversible physics is neglected

Conservation laws : kinematic vs dynamical

- Kinematic conservation law : true for any equation of motion (e.g. Hydrostatic vs non-hydrostatic)

$$\mathcal{M} = \int \rho \mathrm{d}x \mathrm{d}y \mathrm{dz}$$

$$\frac{Ds}{Dt} = 0$$

$$\frac{Dq}{Dt} = 0$$

- Dynamical conservation law : depends on the equation of motion ...

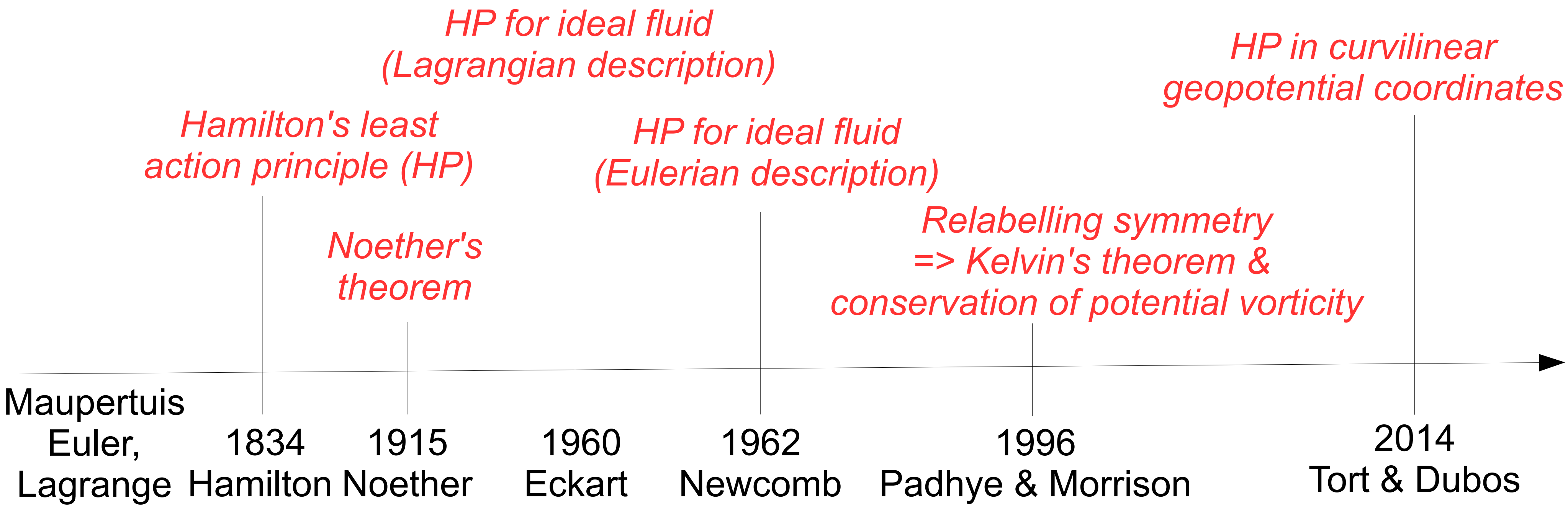
$$\mathcal{L} = \int a^2 \cos^2 \Phi \left(\Omega + \dot{\lambda} \right) \mu \mathrm{d}\lambda \mathrm{d}\phi \mathrm{dz}$$

$$\mathcal{H} = \int \left(\frac{u^2 + v^2 + w^2}{2} + gz + e\left(\frac{1}{\rho},\ s,\ r \right) \right) \rho \mathrm{d}x \mathrm{d}y \mathrm{dz}$$

Kinematic

Dynamic

Back to the origin of dynamical conservation laws

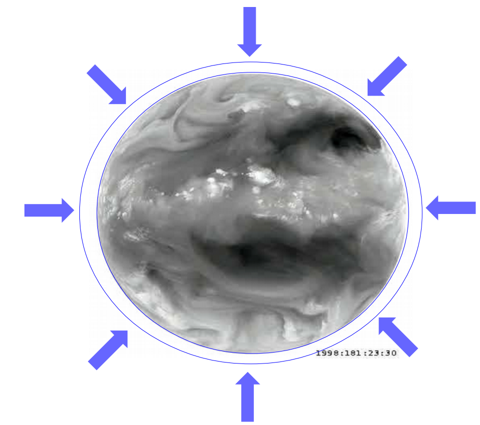


Lagrangian least action principle for fluid flow

(Eckart, 1960 ; Morrison, 1998)

inertia Coriolis pressure gravity

$$\frac{D\dot{\mathbf{x}}}{Dt} + 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$



$$\boxed{\begin{gathered}\delta \int \mathcal{L}\mathrm{d}t = 0 \\ \Updownarrow \\ \frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) + \frac{\partial L}{\partial \mathbf{x}}\end{gathered}}$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s) = \frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}}{2} + (\boldsymbol{\Omega}\times\mathbf{x})\cdot\dot{\mathbf{x}} - gz - e\left(\frac{1}{\rho}, s\right)$$

$$\mathcal{L} = \mathcal{K} + \mathcal{C} - \mathcal{P} = \int L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s)\mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2}\int \dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int (\boldsymbol{\Omega}\times\mathbf{x})\cdot\dot{\mathbf{x}}\mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int \left(e\left(\frac{1}{\rho}, s\right) + \Phi(\mathbf{x})\right)\mathrm{d}m$$ Internal energy
Potential energy

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s) = \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{2} + (\boldsymbol{\Omega} \times \mathbf{x}) \cdot \dot{\mathbf{x}} - gz - e\left(\frac{1}{\rho}, s\right)$$

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) + \frac{\partial L}{\partial \mathbf{x}}$$

$$\mathbf{v} = \frac{\partial L}{\partial \dot{\mathbf{x}}}$$ conjugate momentum = absolute velocity

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) - \nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) + \mathrm{div}\ [\rho \dot{\mathbf{x}} \otimes \mathbf{v}] = \rho \frac{\partial L}{\partial \mathbf{x}}$$

Flux-form momentum budget

$$E = \dot{\mathbf{x}} \cdot \mathbf{v} - L$$
energy

$$\frac{\partial}{\partial t}(\rho E) + \mathrm{div}\ \left[\left(E - \rho \frac{\partial L}{\partial \rho}\right)\rho \dot{\mathbf{x}}\right] = 0$$

Flux-form energy budget

$$B = \mathbf{v} \cdot \dot{\mathbf{x}} - \rho \frac{\partial L}{\partial \rho} - L$$ Bernoulli function

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\nabla \times \mathbf{v}}{\rho} \times \rho \mathbf{u} + \nabla B - T\nabla s = 0$$

Crocco's theorem = curl-form

$$q = \frac{1}{\rho}(\nabla \times \mathbf{v}) \cdot \nabla s$$

$$\frac{Dq}{Dt} = 0$$

Potential vorticity

Conservation laws for continous models

What they are

- Lagrangian vs integral
- Adiabatic vs diabatic
- Robust vs fragile

Where they come from

- Kinematics vs dynamics
- Variational principles, symmetries and dynamical conservation laws

What they imply

- stability

$$\int f(q,s)\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Casimir (kinematic) invariants

$$\mathcal{H} = \int \left(\frac{u^2+v^2+w^2}{2} + gz + e\left(\frac{1}{\rho},\, s,\, r\right)\right)\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Energy

$$\mathcal{L} = \int a^2 \cos^2\Phi \left(\Omega + \dot{\lambda}\right)\mu \mathrm{d}\lambda \mathrm{d}\phi \mathrm{d}z$$

Absolute angular momentum (AAM)

Conservation laws for continous models : what they imply

Energy-Casimir stability theory

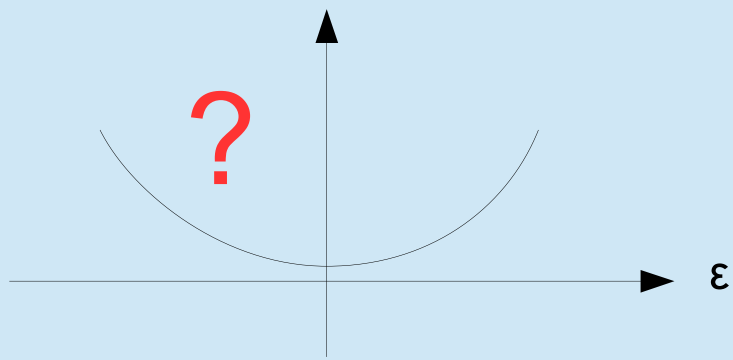
- consider some integral invariant

$$\mathcal{E} = \mathcal{H} - \int f(s)\rho \mathrm{d}x\mathrm{d}y\mathrm{z}$$

pseudo-energy

- suppose that

$$\mathcal{E}\left[\mathbf{u} + \varepsilon\delta\mathbf{u},\ \rho + \varepsilon\delta\rho,\ s + \varepsilon\delta s\right]$$



=> stationary and *stable* flow !

$$\delta\mathcal{E} = 0$$

Identify steady flow

$$\delta^2\mathcal{E} > 0$$

Sufficient stability criterion

Conservation laws for continous models : what they imply

Energy-Casimir stability theory

$$\mathcal{E} = \mathcal{H} - \int f(s)\rho \mathrm{d}x\mathrm{d}y\mathrm{z}$$

pseudo-energy

$$\begin{aligned}\delta\mathcal{E} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int \left(\frac{1}{2}\left(\mathbf{u}+\varepsilon\delta\mathbf{u}\right)^2 + gz + e - f\right)\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int \mathbf{u}\cdot\delta\mathbf{u}\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

$$\mathbf{u} = 0$$

$$\begin{aligned}\delta\mathcal{E} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int \left(gz + e\left(\rho+\varepsilon\delta\rho,\ s\right) - f(s)\right)\left(\rho+\varepsilon\delta\rho\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int \left(gz + e + \frac{p}{\rho} - f\right)\delta\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

$$gz + h = f(s)$$

Montgomery potential

$$T = \frac{\mathrm{d}f}{\mathrm{d}s}$$

$$\begin{aligned}\delta\mathcal{E} &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int \left(gz + e\left(\rho,\ s+\varepsilon\delta s\right) - f\left(s+\varepsilon\delta s\right)\right)\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int \left(T - \frac{\mathrm{d}f}{\mathrm{d}s}\right)\delta s\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z\end{aligned}$$

$$\Rightarrow \quad g + \frac{1}{\rho}\partial_z p = 0$$

$$\delta\mathcal{E} = 0$$

Hydrostatic balance

$$\delta^2\mathcal{E} > 0$$

Static stability ds/dz > 0

Conservation laws for continous *and discrete* models

More conserved integral invariants,
more guarantees of stability

$$\int f(q,s)\rho \mathrm{d}x\mathrm{d}y\mathrm{dz}$$

Casimir (kinematic) invariants

$$\mathcal{H} = \int \left(\frac{u^2+v^2+w^2}{2} + gz + e\left(\frac{1}{\rho},\ s,\ r\right)\right)\rho \mathrm{d}x\mathrm{d}y\mathrm{dz}$$

Energy

$$\mathcal{L} = \int a^2 \cos^2\Phi \left(\Omega + \dot{\lambda}\right)\mu \mathrm{d}\lambda\mathrm{d}\phi\mathrm{dz}$$

Absolute angular momentum (AAM)

Context : Cartesian geometry

- Conservation laws for continuous models
- **From the continuous to the discrete, and what you lose**
- Numerical dispersion : illustration with the rotating shallow-water equations

Discrete degrees of freedom

- Discrete representations of scalar and vector fields
- Interpolation/reconstruction/projection ; order of accuracy
- Immediate consequences : (non-)preservation of algebraic identities

We assume that the following has already been done :

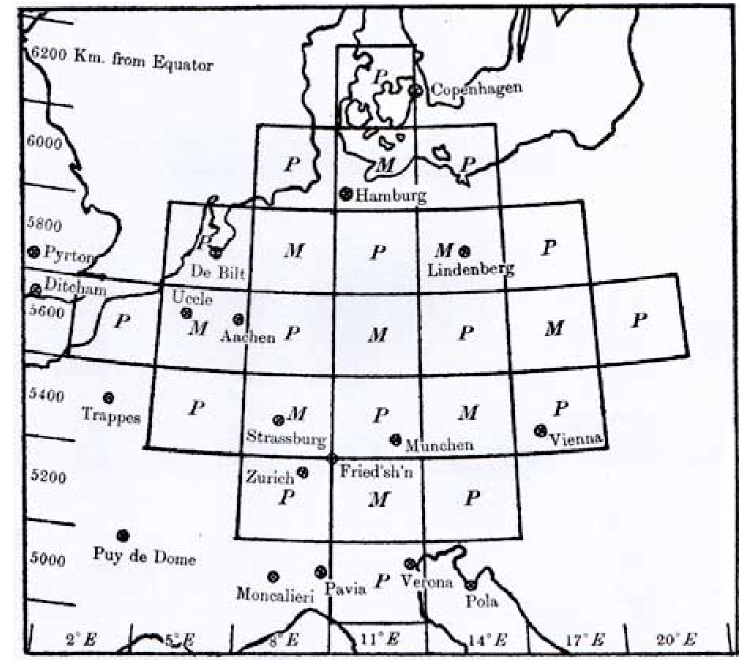
- A set of equations has been chosen :
  thermodynamics, geometry, hydrostatic/NH/Boussinesq
- A flow representation has been chosen : Eulerian vs non-Eulerian vertical coordinate
- Prognostic variables have been chosen
- It is known how to obtain diagnostic variables : local or non-local (e.g. elliptic) problem

Now we want to represent each field (prognostic, diagnostic) by a large but finite set of numbers.

- How to do it ? - basic examples
- How to apply operations (algebraic, differential) to these discrete representations ?
- Consequences for the properties of the scheme ?

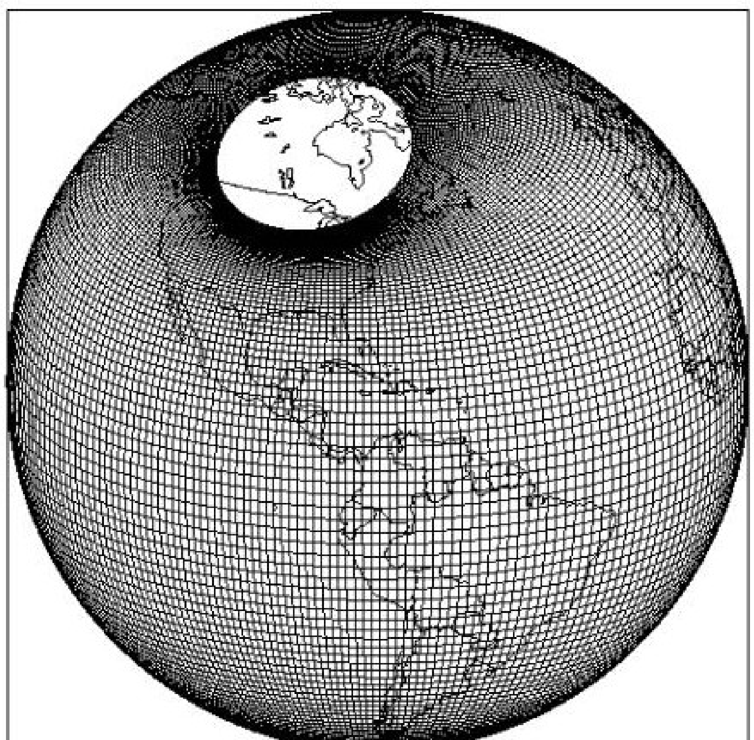
Simplest idea :

- Sample field(s) at a number of points
- Those points usually form a *mesh*
- *Interpolate* between these points as needed

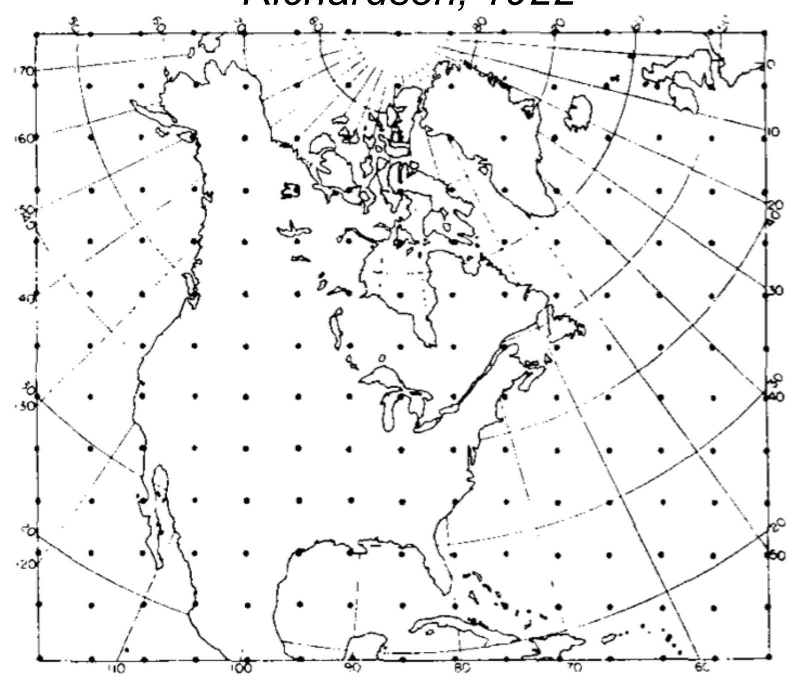


MAP OF POINTS FOR PRESSURE (P) AND MOMENTUM (M) USED IN THE EXAMPLE of Ch. 9.

Richardson, 1922



POP



Charney et al., 1950

Interpolation is a *reconstruction* problem :

- Data : M values of function f at points xi
- Unknown : function g in D-dimensional space with the same values as f at xi
- If D>M : over-conditioned problem => least-squares
- Once g has been found, use it instead of f to compute values and derivatives where needed

Local interpolation *=> Finite differences*		*Global interpolation* *=> Pseudo-spectral*
Solve N independent interpolation problems, using a small number of data around each point i (stencil)		use all points : M=N=D
Linear (d=1) / Quadratic (d=2)	*Example*	Fourier
Error for values ~ dx^(d+1) Error for gradient ~ dx^d	*Accuracy* *(Error f-g)*	Error ~ exp(-cN) *if f very smooth*
Small	*Cost*	*OK to very large*
Good	*Parallelism*	*Poor*

Geophysical flow fields not very smooth : high-order (>4) interpolation probably not worth the cost (see Mavriplis, 2011)

Typical spaces for g :

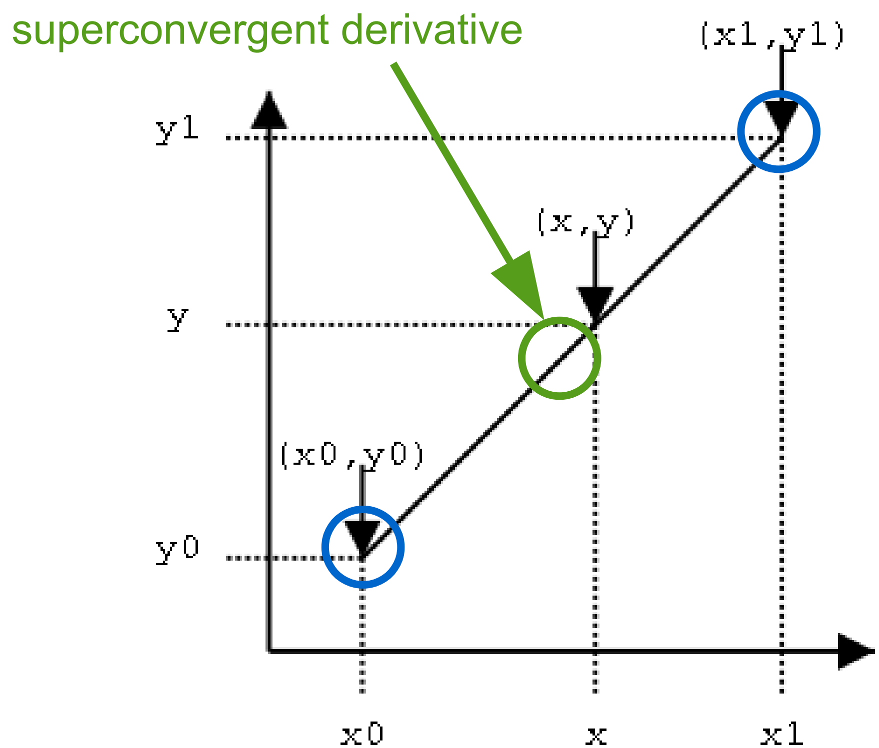
- Polynomials
- Splines
- Radial basis functions (RBF)
- ...

Interpolation is a *reconstruction* problem :

- Data : M values of function f at points xi
- Unknown : function g in D-dimensional space with the same values as f at xi
- If M>D : over-conditioned problem => least-squares
- Once g has been found, use it instead of f to compute values and derivatives where needed

- 2 data enough for 1D linear interpolation
- Interpolated values 2nd-order accurate
- gradient 1st-order accurate

- except at midpoint : second-order accurate gradient
- superconvergence : extra accuracy at some special points



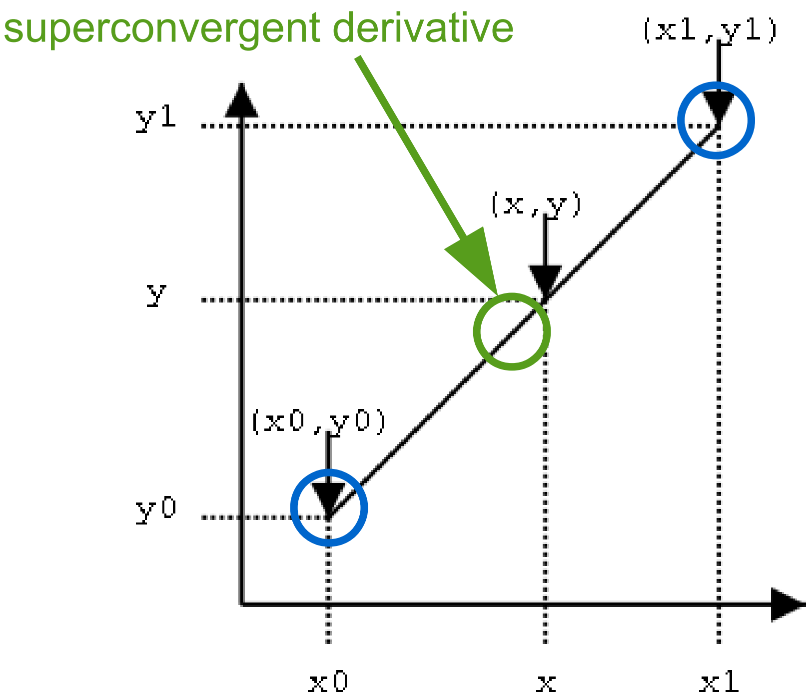
Linear interpolation

Simplest idea : interpolation

- Sample field(s) at a number of points
- Those points usually form a *mesh*
- *Interpolate* between these points as needed

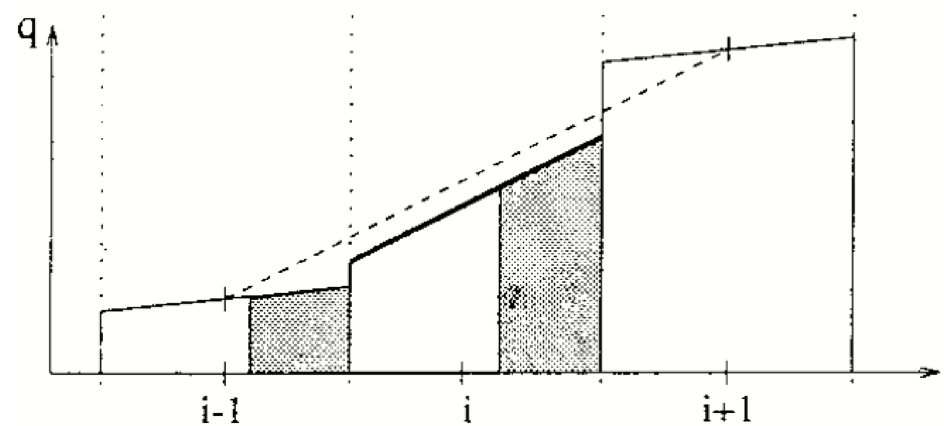
Another idea : « histopolation »

- Data = field integrated over each mesh cells
- Use cell data and nearest neighbors to *reconstruct* field
- Typical of *finite volume methods*



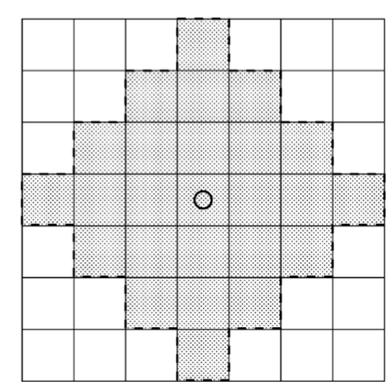
Linear interpolation

=> very good accuracy/cost ratio using regular Cartesian meshes and superconvergence

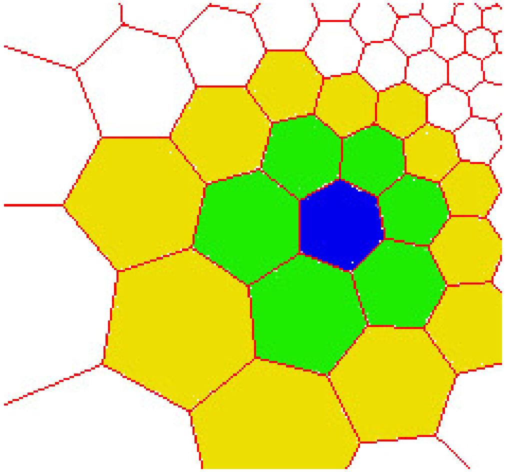


Hourdin & Armengaud, 1999

Stencil for 4th-order histopolation (Ullrich et al., 2010)



Multidimensional reconstruction



D		
1	Constant	1
3	Linear	$x\ y$
6	Quadratic	$x^2\ xy\ y^2$
10	Cubic	$x^3\ x^2y\ xy^2\ y^3$
15	Quartic	

3D

D		
1	Constant	1
4	Linear	$x\ y\ z$
10	Quadratic	$x^2\ y^2\ z^2\ xy\ xz\ yz$
20	Cubic	$x^3\ y^3\ z^3\ x^2y\ x^2z\ y^2x\ y^2z\ z^2x\ z^2y\ xyz$

for order-n reconstruction

1D : $D \geq n$
2D : $D \geq n(n+1)/2$
3D : $D \geq (d+1)^3/3$

N local DxD linear systems to solve (pre-computation)
Store NxD weights

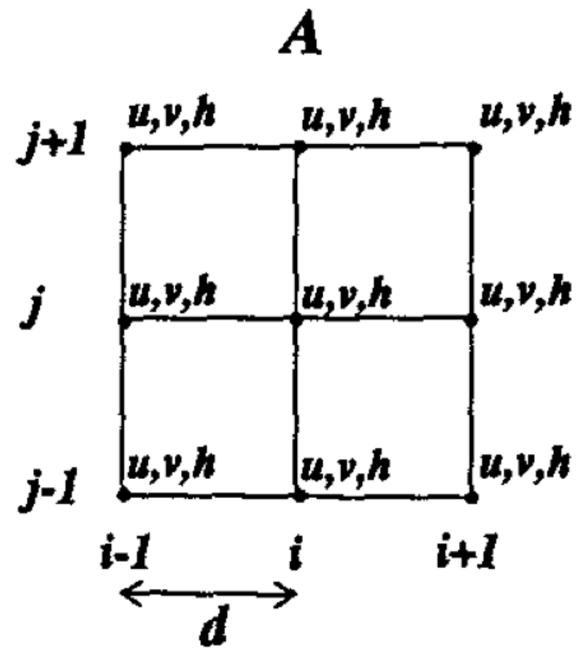
High-order reconstruction rapidly very costly !

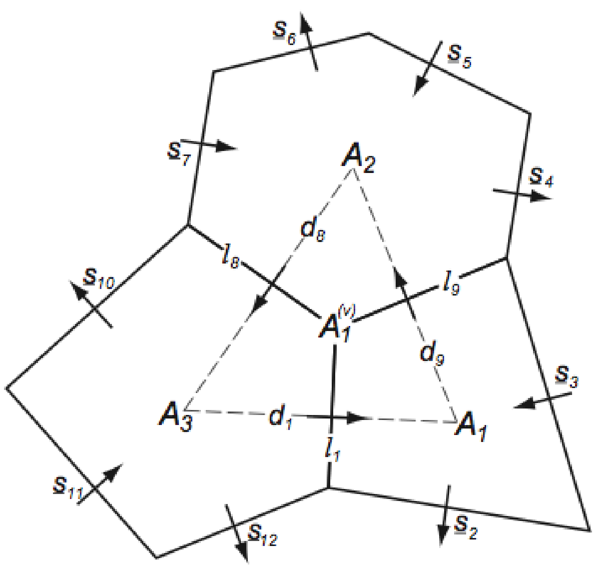
Cartesian or 2Dx1D meshes more efficient

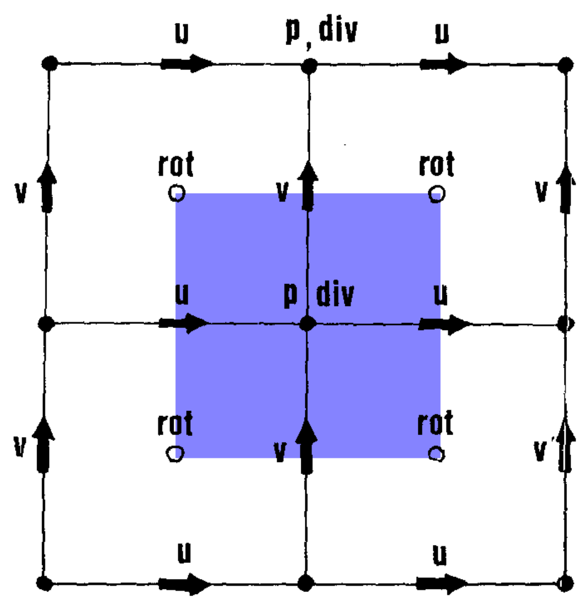
Representation of vector fields

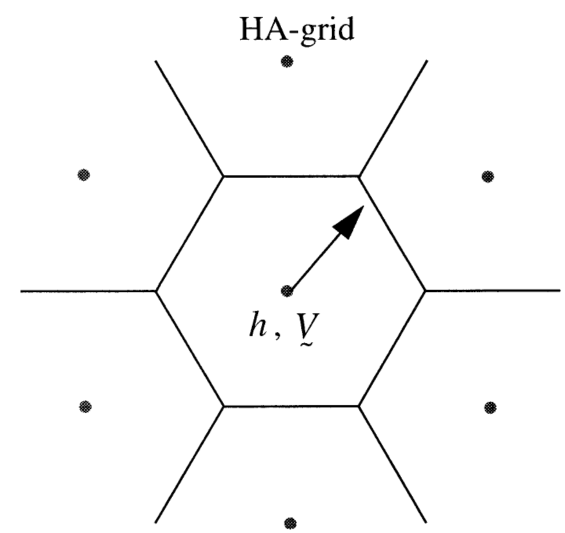
- components in coordinate system
- components on local basis
- components normal/tangential to mesh faces/edges
- curl / div

- Pointwise value (finite difference)
- Cell-integrated value (finite volume)







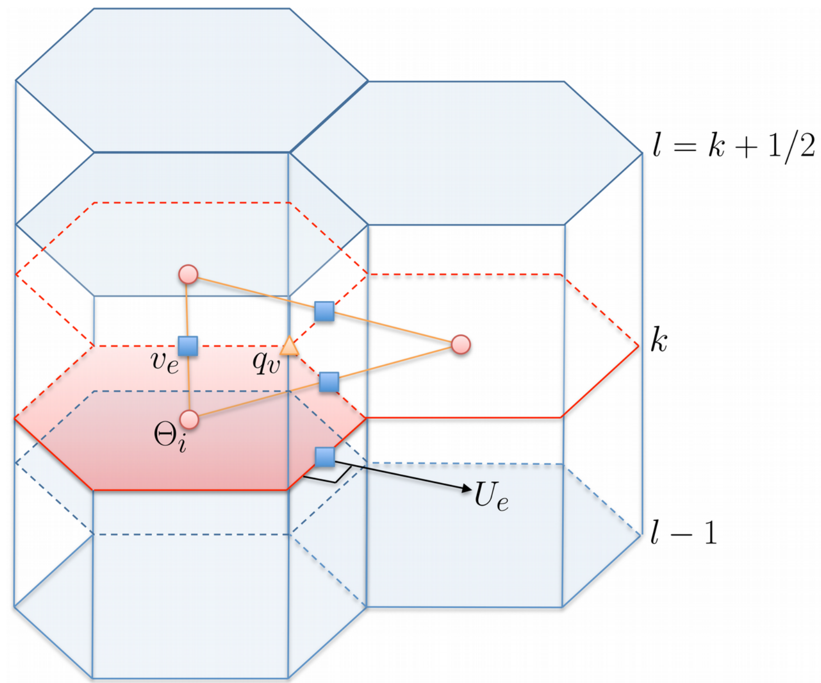


Different quantities possibly placed at different places : *staggered meshes*

Representation of vector fields

Strong recent trend in various fields of computational physics : *discrete differential geometry (see Thuburn & Cotter, 2012)*

- Describe fields through their integrals over geometric objects
- Match geometric object (point, line, surface, volume) and « nature » of field



• Scalar field	pointwise value	0D	scalar field
• Momentum	line integral	1D	covariant
• Flux	integral across surface	2D	contravariant
• Density	integral over cell	3D	mass-weighted scalar field

Representation of scalars and vectors : implications

No shortage of discrete representations of scalar and vector fields

- Which algebraic identities survive/fail at the discrete level ?
- How to analyse the behavior that *emerges* from a certain combination of choices ?

$$\nabla f(a,b) = \frac{\partial f}{\partial a}\nabla a + \frac{\partial f}{\partial b}\nabla b$$

usually ***fails***

$$\mathrm{div}\,(f\mathbf{u}) = \mathbf{u}\cdot\nabla f + f\mathrm{div}\;\mathbf{u}$$

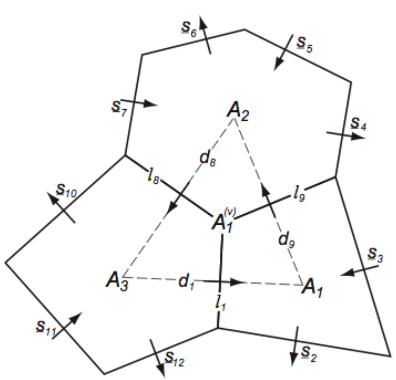
usually ***fails … but***

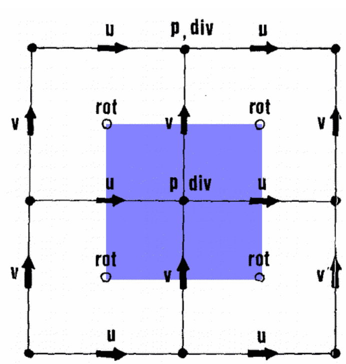
$$\int \mathbf{u}\cdot\nabla f + \int f\mathrm{div}\;\mathbf{u} = 0$$

often ***doable :***
compatible discretizations of grad and div
Taylor, 2010

$$\int_\Omega \mathrm{div}\;\mathbf{F}d\Omega = \int_{\partial\Omega}\mathbf{F}\cdot d\mathbf{\Sigma}$$

often **works** : with flux-form, yields
conservation of linear integral invariants
Finite-volume approach





Enough for conservation of non-linear invariants ? More later ...

Which algebraic identities survive/fail at the discrete level ?

$$\int_\Omega \mathrm{div}\ \mathbf{F} \mathrm{d}\Omega = \int_{\partial\Omega} \mathbf{F} \cdot \mathrm{d}\mathbf{\Sigma}$$

$$\int_\Sigma (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{\Sigma} = \oint_{\partial\Sigma} \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

$$\int_{AB} \nabla f \cdot \mathrm{d}\mathbf{l} = f(B) - f(A)$$

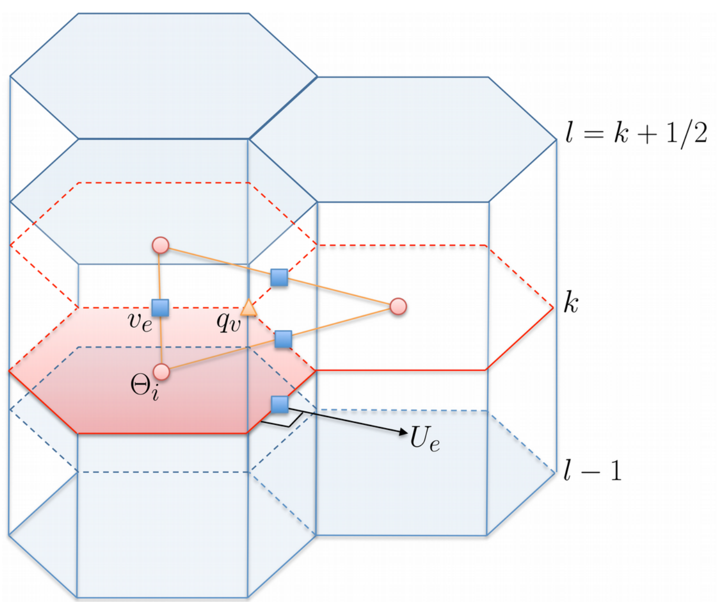
$$\nabla \times (\nabla f) = 0$$

$$\mathrm{div}\,(\nabla \times \mathbf{v}) = 0$$

often **works** : together with flux-form, yields *conservation of linear invariants*
Finite-volume approach

often easy using discrete differential geometry

important for vorticity dynamics



Context : Cartesian geometry

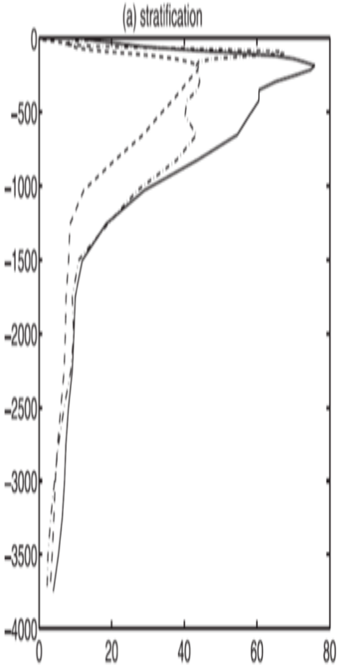
- Conservation laws for continuous models
- From the continuous to the discrete, and what you lose
- **Numerical dispersion : illustration with the rotating shallow-water equations**

Why analyze the RSW equations rather than the full 3D equations ?

- Simpler
- Disentangle issues related to vertical and horizontal discretization
- 3D hydrostatic modes separate into vertical structure equation and horizontal propagation problem

Basic idea : wave propagation reveals how the different fields are coupled to each other

- Hopefully numerical wave propagation resembles physical wave propagation …
- Many bad surprises ahead !
- If equations include many effects (forces), hard to have all couplings right
- Damage control : do at least the fast waves well

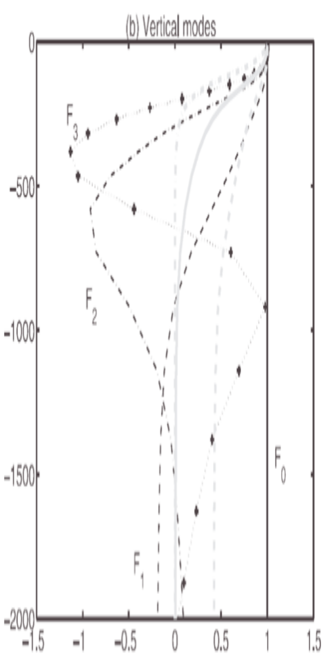


Bouss. + Coriolis + hydrostat.

$$
\begin{aligned}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} &= 0\\
\frac{Du}{Dt}-f(y)v+\frac{1}{\rho_r}\frac{\partial p}{\partial x} &= 0\\
\frac{Dv}{Dt}+f(y)u+\frac{1}{\rho_r}\frac{\partial p}{\partial y} &= 0\\
\frac{1}{\rho_r}\frac{\partial p}{\partial z} &= b-g,\\
\frac{Db}{Dt} &= 0.
\end{aligned}
$$

Linéarisation autour du repos

$$
\begin{aligned}
\frac{\partial \tilde u}{\partial x}+\frac{\partial \tilde v}{\partial y}+\frac{\partial \tilde w}{\partial z} &= 0,\\
\frac{\partial \tilde u}{\partial t}-f(y)\tilde v+\frac{1}{\rho_r}\frac{\partial \tilde p}{\partial x} &= 0\\
\frac{\partial \tilde v}{\partial t}+f(y)\tilde u+\frac{1}{\rho_r}\frac{\partial \tilde p}{\partial y} &= 0\\
\frac{1}{\rho_r}\frac{\partial}{\partial t}\frac{\partial \tilde p}{\partial z}+N^2(z)\tilde w &= 0.
\end{aligned}
$$



Solution séparable

$$\tilde w = W(z)\,w'(x,y,t), \qquad \tilde p = \mathrm{d}W/\mathrm{d}z\,\rho_r p'(x,y,t),$$
$$\tilde u = \mathrm{d}W/\mathrm{d}z\,u'(x,y,t), \qquad \tilde v = \mathrm{d}W/\mathrm{d}z\,v'(x,y,t)$$

Structure verticale

$$\frac{1}{N^2}\frac{\mathrm{d}^2 W}{\mathrm{d}z^2}+\lambda W = 0$$

Valeur propres inconnues $\lambda_n = 1/gH_n$

$$w' = \frac{1}{gH_n}\frac{\partial p'}{\partial t} \qquad \Rightarrow$$

Dynamique horizontale

$$
\begin{aligned}
\frac{\partial p'}{\partial t}+gH_n\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}\right) &= 0\\
\frac{\partial u'}{\partial t}-f(y)v'+\frac{\partial p'}{\partial x} &= 0\\
\frac{\partial v'}{\partial t}+f(y)u'+\frac{\partial p'}{\partial y} &= 0
\end{aligned}
$$

RSW : exact dispersion relationship

$$
\begin{aligned}
\partial_t p' + c^2 \left(\partial_x u' + \partial_y v' \right) &= 0 \\
\partial_t u' - f v' + \partial_x p' &= 0 \\
\partial_t v' + f u' + \partial_x p' &= 0
\end{aligned}
$$

Stationary modes

$$
\begin{aligned}
- f v' + \partial_x p' &= 0 \\
f u' + \partial_y p' &= 0
\end{aligned}
$$

Geostrophic balance

$$
\begin{aligned}
\delta' &= \partial_x u' + \partial_y v' \\
\zeta' &= \partial_x v' - \partial_y u'
\end{aligned}
$$

$$
\begin{aligned}
\partial_t p' + c^2 \delta' &= 0 \\
\partial_t \delta' - f\zeta' + \left(\partial_{xx} + \partial_{yy} \right) p' &= 0 \\
\partial_t \left(\frac{\zeta'}{f} - \frac{p'}{c^2} \right) &= 0
\end{aligned}
$$

PV conservation (linearized)

Non-stationary modes carry no PV

Non-stationary modes

$$
\left(\partial_{tt} + f^2 \right) p' - c^2 \left(\partial_{xx} + \partial_{yy} \right) p' = 0
$$

$$
p' \sim \exp i \left(kx + ly - \omega t \right)
$$

$$
K^2 = k^2 + l^2
$$

$$
\omega^2(k) = f^2 + K^2 c^2
$$

Inertia Gravity

Recap : waves in 3D stratified Boussinesq

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}$$

Vertical density profile (stratification) determines N(z), eigenvalues c^{-2} and baroclinic structure W(z)

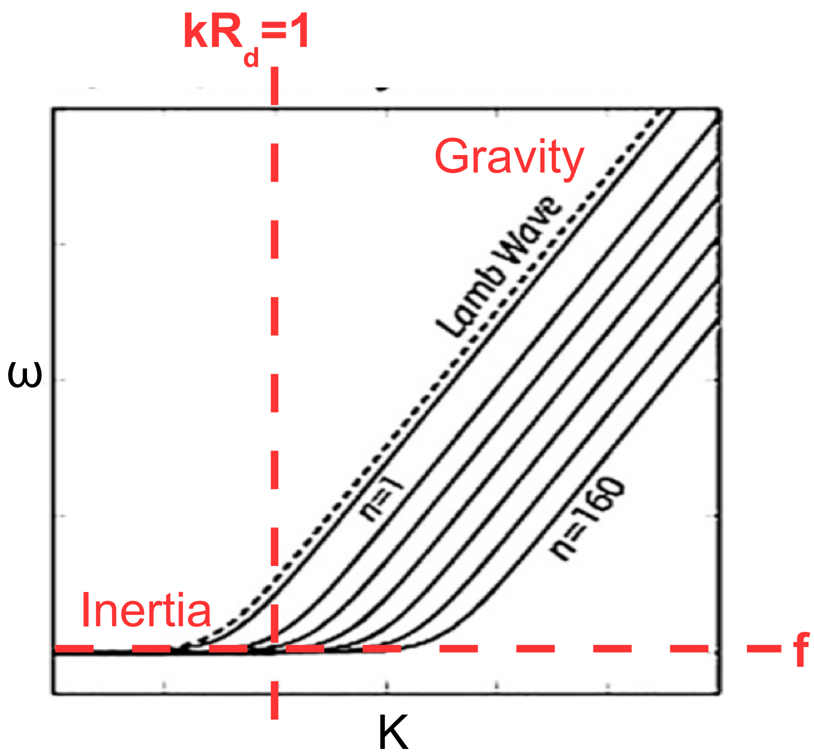
$$N^{-2}(z)\frac{\mathrm{d}W}{\mathrm{d}z^2} + c^{-2}W = 0$$

$$\begin{aligned} \partial_t p' + c^2(\partial_x u' + \partial_y v') &= 0 \\ \partial_t u' - f v' + \partial_x p' &= 0 \\ \partial_t v' + f u' + \partial_x p' &= 0 \end{aligned}$$

$$p' \sim \exp i(kx + ly - \omega t)$$

$$\omega^2(k) = f^2 + K^2 c^2 \qquad K^2 = k^2 + l^2$$

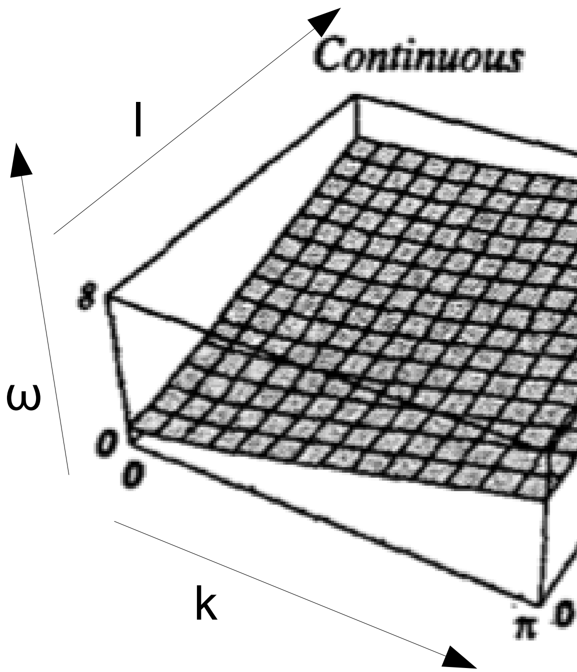
Rossby radius R_d=c/f

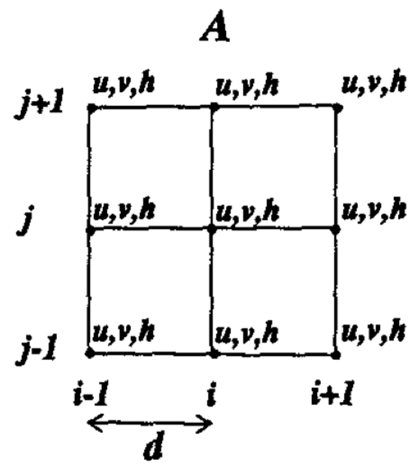


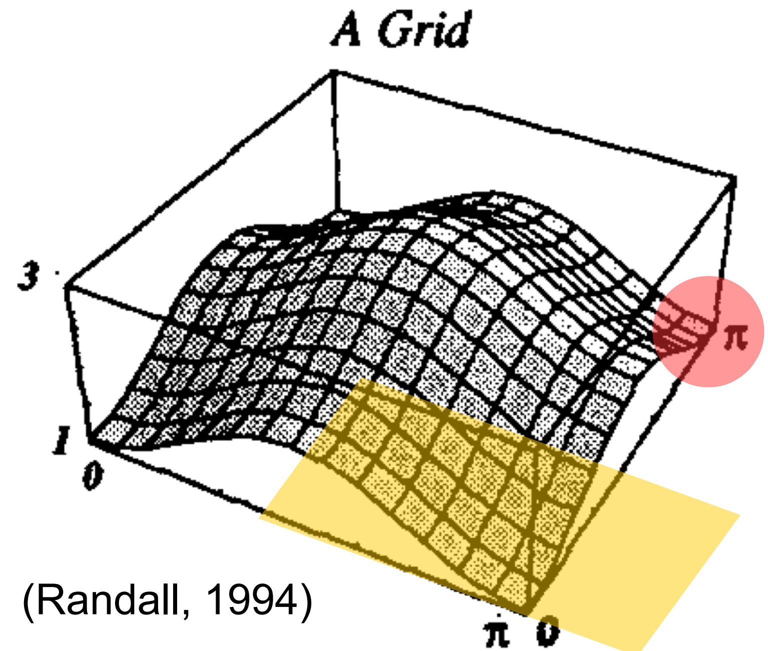
Do the waves propagate numerically as in the continuous model ?

- Numerical integrations in time
- Look for eigenmodes and frequencies
  - Regular mesh (Cartesian / Hexagonal / Triangular) : Fourier transform => analytic dispersion relationship (*Randall, 1994*)
  - Unstructured mesh : numerical (*Weller et al., 2012*)

RSW : numerical dispersion relationship

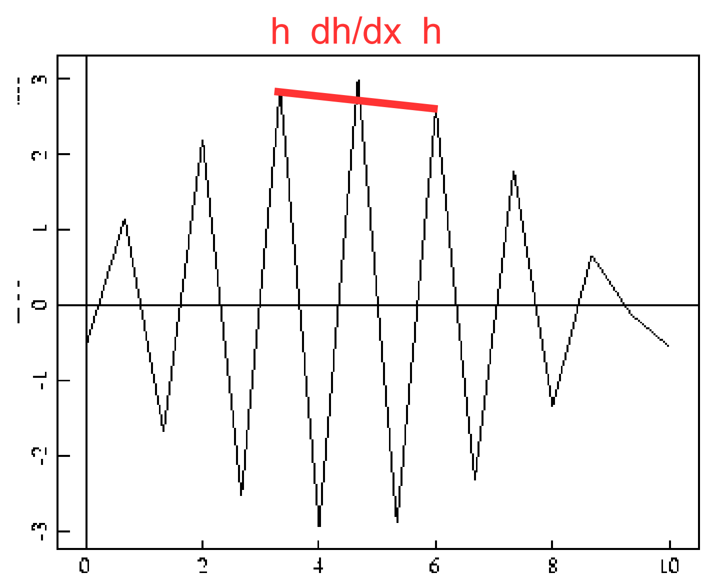






(Randall, 1994)

- Stationary checkerboard pattern
- « numerical mode » or « spurious mode »
- Wrong sign of group velocity



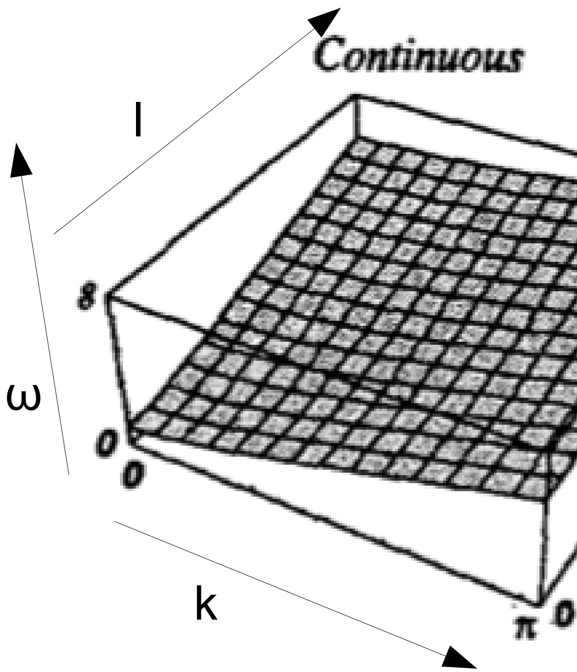
Reason :

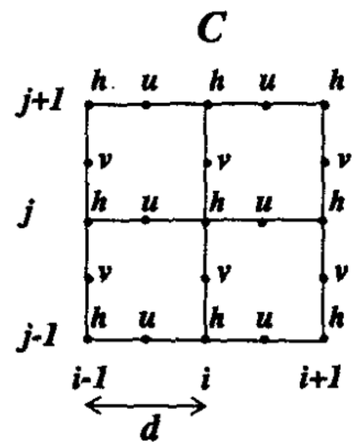
- u and du/dx are *collocated*
- the gradient of an alternating pattern is 0 instead of being large
- Frequency of small-scale waves underestimated

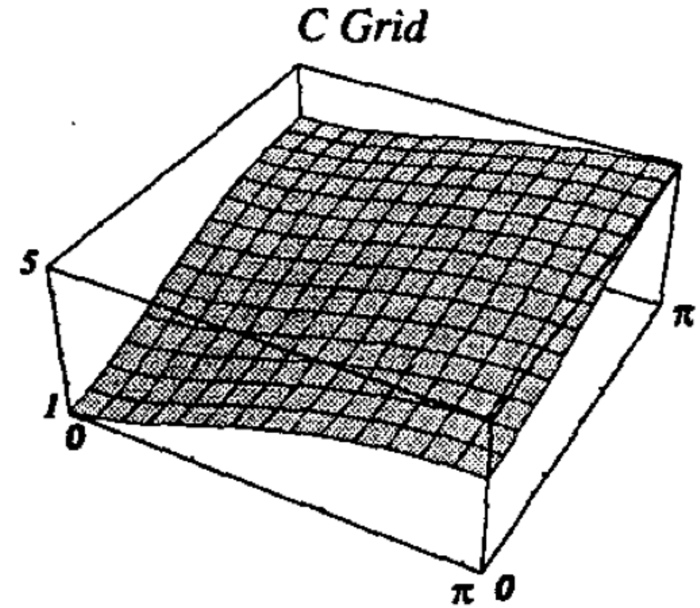
Usually better to *stagger* a field and its derivative
Or dissipate high spatial frequencies (but reduces effective model resolution)

RSW : numerical dispersion relationship

Usually better to *stagger* a field and its derivative => C-grid



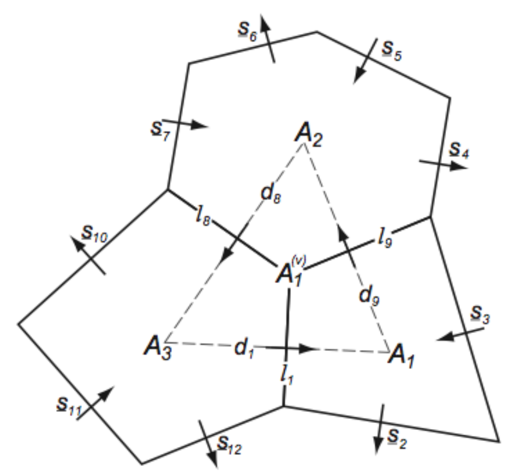




(Randall, 1994)

- Looks much better
- Gravity-waves propagate physically due to velocity – mass staggering

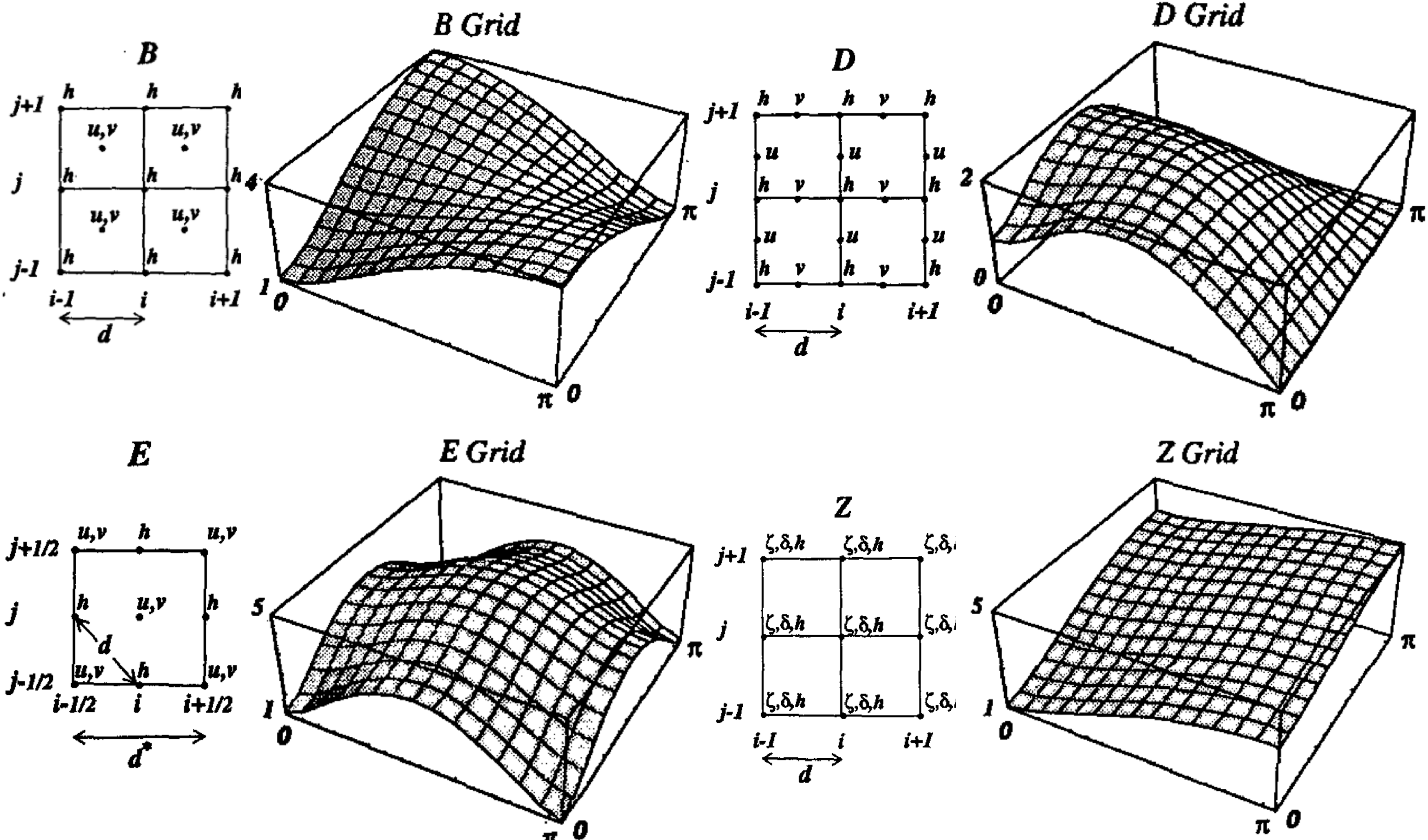
- But:
  - u and v at different sites
  - Coriolis term requires averaging
  - Rossby radius unresolved => Inertial waves too slow



C-grid staggering on unstructured mesh

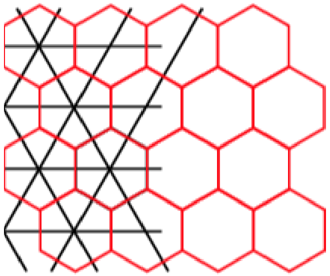
RSW : numerical dispersion relationship

Gravity waves want u-v staggered ; inertial waves want u-v collocated => many possible trade-offs



Excellent dispersion but must solve Poisson problems

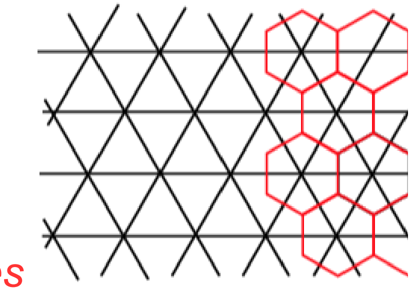
Regular polygonal C-grids



Hexagonal C-grid
Mass N
Div N
Vort 2N

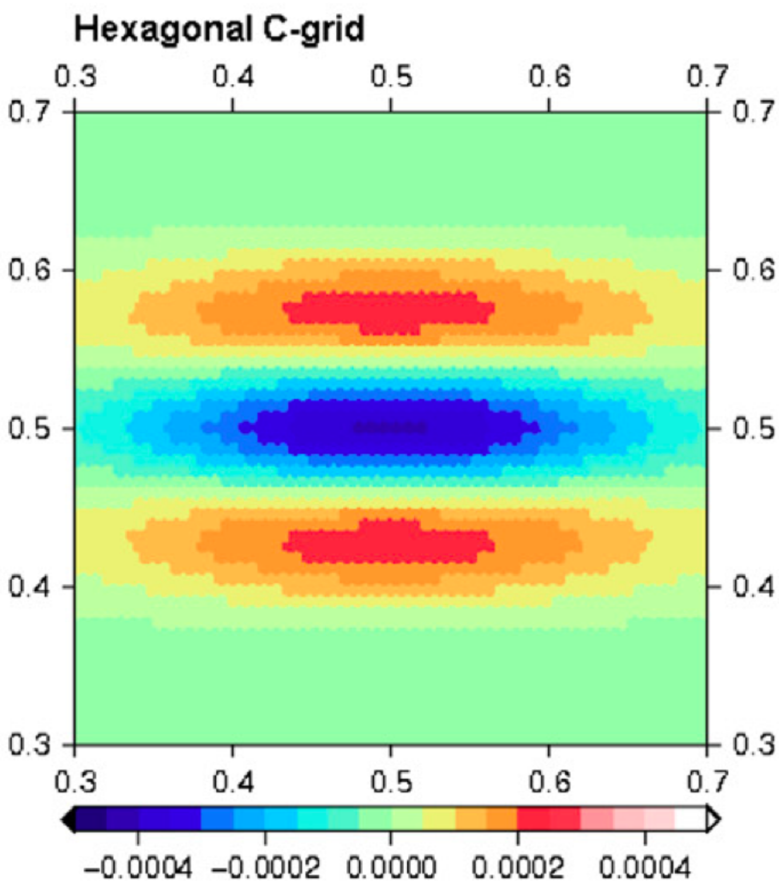
Cartesian C-grid
Mass N
Div N
Vort N

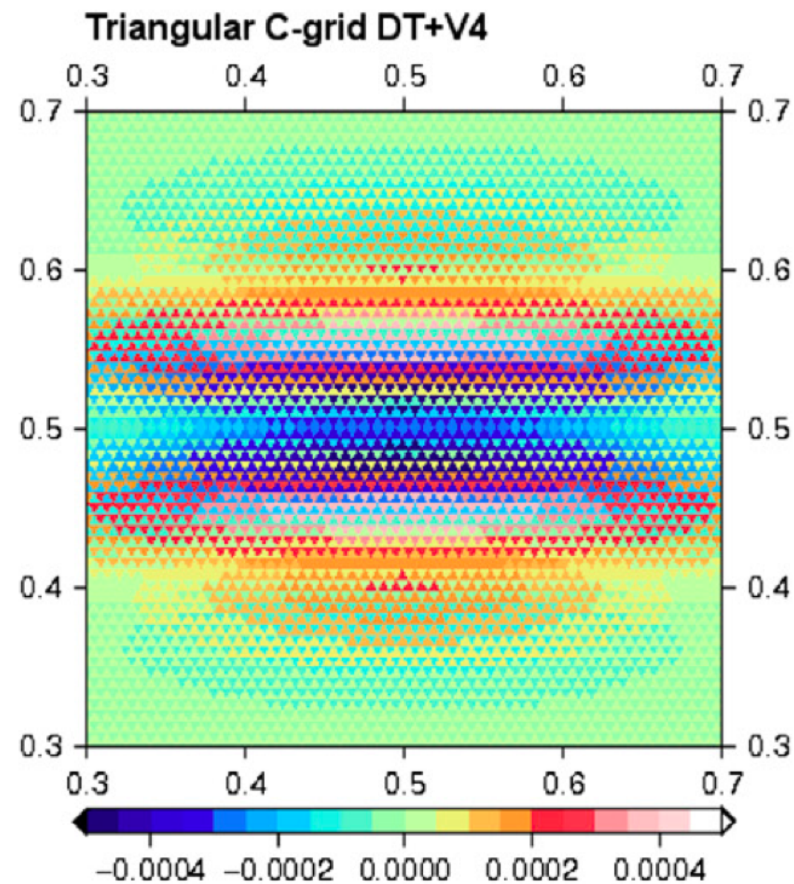
Triangular C-grid
Mass 2N
Div 2N
Vort N



Vortical numerical modes

Divergent numerical modes





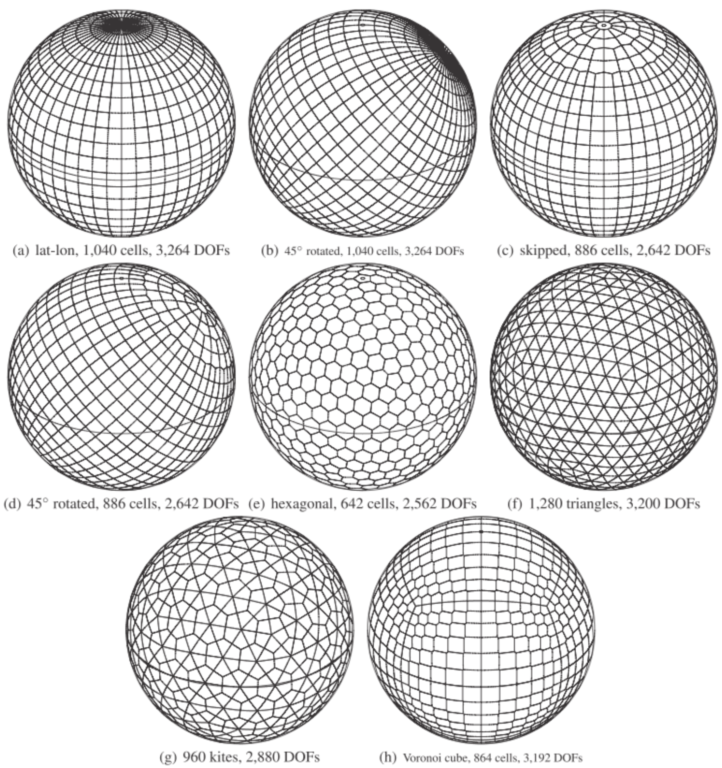
Gassmann., 2011

Fig. 5. Divergences of linear experiments after 50 time steps.

Imbalance in the number of degrees of freedom => numerical *branches* in the dispersion relationship

General polygonal grids : more possibilities, more problems ...

Weller et al., 2012



… and solutions : see Thuburn (2008), Thuburn et al. (2009, 2014), ...

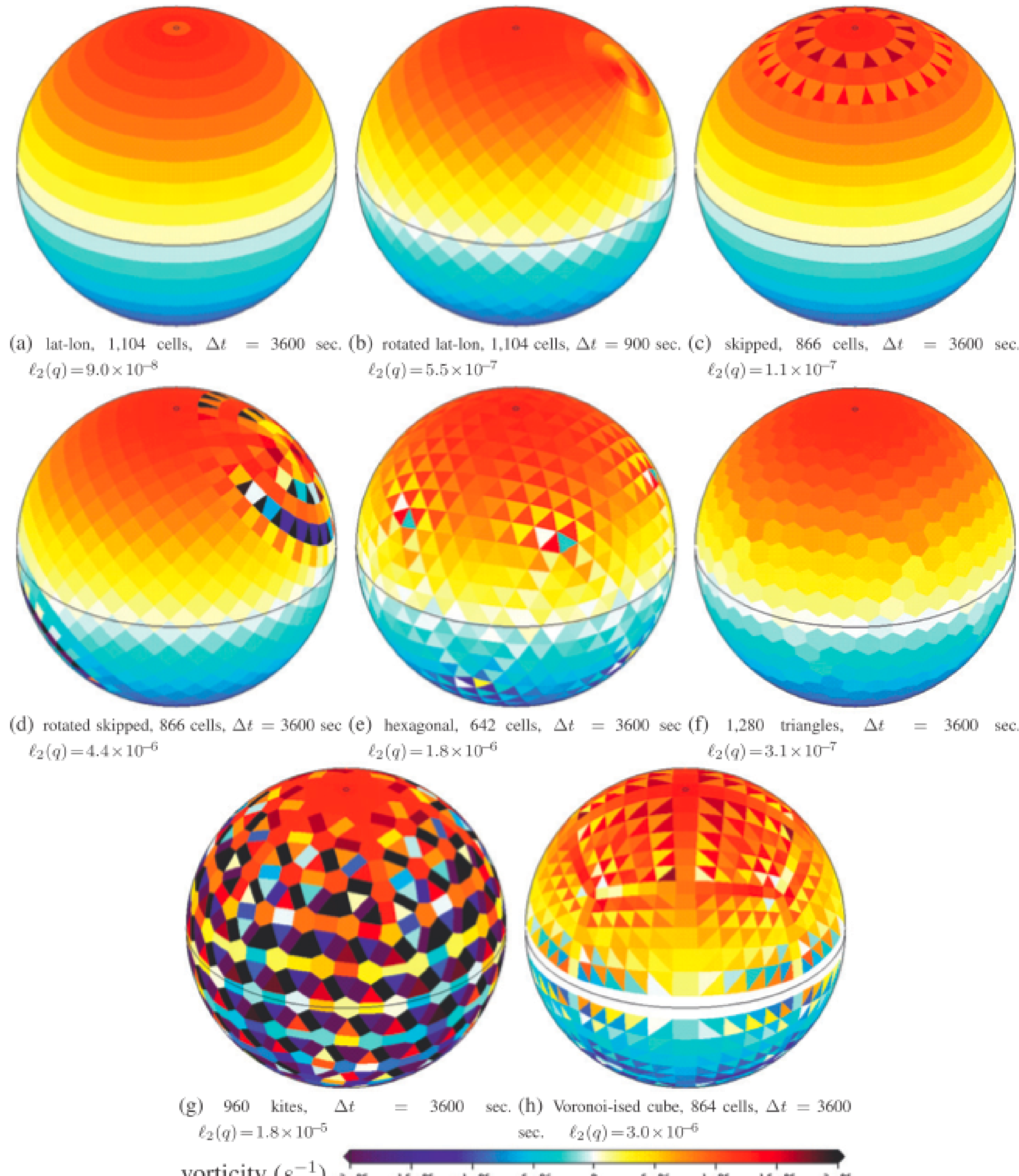


FIG. 7. Relative vorticity on the dual grid after 5 days for the linearized shallow water equations simulating a linearized version of Williamson et al. (1992), test case 2.

Conservation laws

- *Generic* due to variational principle underlying the adiabatic equations of motion
- *Nonlinear* integral invariants guarantee stability close to certain base flows
- *Linear* integral invariants (mass, species) can be preserved by a *finite-volume approach*

From the continuous to the discrete, and what you lose

- Many ways to *represent* prognostic and diagnostic fields
- Determines to a great extent how they *couple* to each other
- Most algebraic identities lost
- Certain important vector-differential identities can be preserved
- Care required to avoid bad surprises with numerical dispersion
  - unphysical *branches*
  - unphysical propagation of small-scale modes
  - propagative => stationary but also stationary => propagative

References

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Morrison (1998)	Reviews of Modern Physics **70**, 467-521
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Thuburn & Cotter (2012)	SIAM J. Sci. Comp. **34**, B203–B225,
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