	NEMO	**ROMS**	**IFS/ARPEGE**	**MesoNH**	**WRF**	**EndGAME**	**LMDZ**	**DYNAMICO**
Geometry	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG	SG+TSA	SG+TSA
Dynamics	HB	HB	FCE	A	FCE	FCE	HPE	HPE/(FCE)
Grid	CC	CC	LL	CC	CC	LL	LL	HEX
Disc. Dyn	FD	FV	SP	FD	FV	FD	FD	FD
Transport	FV	FV	SL	FV	FV	FV	FV	FV
Conserv.	M, E/Z	M		M	M	M	M, E/Z	M, E
Time	Split-EX	Split-EX	SI	EX	Split-HEVI	SI	EX	EX/HEVI
Helmholtz			Direct	Direct		Iter		

SG	Spherical-Geoid
TSA	Traditional Shallow-Atmosphere
FCE	Fully Compressible Euler
HPE	Hydrostatic Primitive Eq.
HB	Hydrostatic Boussinesq
A	Anelastic
EX	Explicit
SI	Semi-Implicit
Split	Split
HEVI	Horizontally Explicit, Vertically Implicit
Direct	Direct (spectral)
Iter	Iterative

CC	Cartesian Curvilnear
LL	Latitude-Longitude
HEX	Icosahedral-Hexagonal
FD	Finite Difference
FV	Finite Volume
FE	*Finite Element*
SE	*Spectral Element*
SL	Semi-Lagrangian
SP	Spectral
M	Mass and scalars
E	Energy
Z	Enstrophy

Discretization IV

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What do you have to say about a numerical model/scheme ?

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- All ? Independently ? Incompatibilities ?
- Implications for performance / parallelism ?

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Context : Spherical shell geometry

- Vertical discretization : staggering, numerical dispersion
- Horizontal mesh : pole problem, spectral method, quasi-uniform meshes
- Conservation of non-linear integral invariants : from clever solutions to systematic approaches

Vertical discretization :

- **Vertical coordinates**
- **Where should we place prognostic/diagnostic variables**
- **Criteria : mass/transport consistency, numerical dispersion**

Spherical meshes

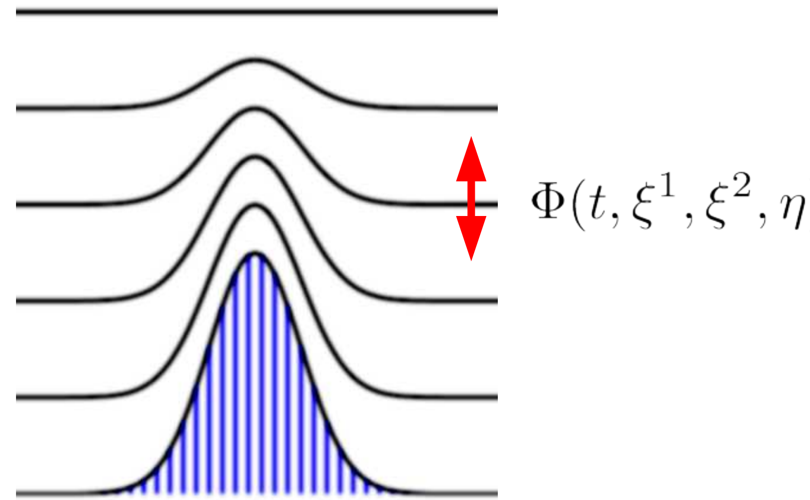
- curvilinear Cartesian meshes
- global curvilinear Cartesian meshes : the pole problem
- a meshless method : spectral method
- quasi-uniform meshes and associated issues

Conservation of non-linear integral invariants

- a few clever solutions
- towards systematic approaches

Recap : hydrostatic dynamics, generalized vertical coordinates & prognostic variables

- a hydrostatic adjustment occurs at each time step
- altitude z should be *diagnostic*
- vertical coordinate should be non-Eulerian



kinematics

dynamics

Hybrid mass-based coordinate

- *Diagnose pseudo-density mu from total column mass M*
- *Prognose M*
- *Diagnose dmu/dt*
- *Diagnose eta_dot*
- *Prognose entropy*

- Hydrostatic adjustment => geopotential
- Prognose momentum

Lagrangian coordinate

- *Prognose pseudo-density mu*
- *Prognose entropy*
- *If needed, vertical remap*

- Hydrostatic adjustment => geopotential
- Prognose momentum

Generalized vertical coordinates & prognostic variables

Hydrostatic
Boussinesq
Anelastic

4+1 fields
Redundant flow description

3+1 fields
Non-redundant flow description

Euler

6 fields
Redundant flow description

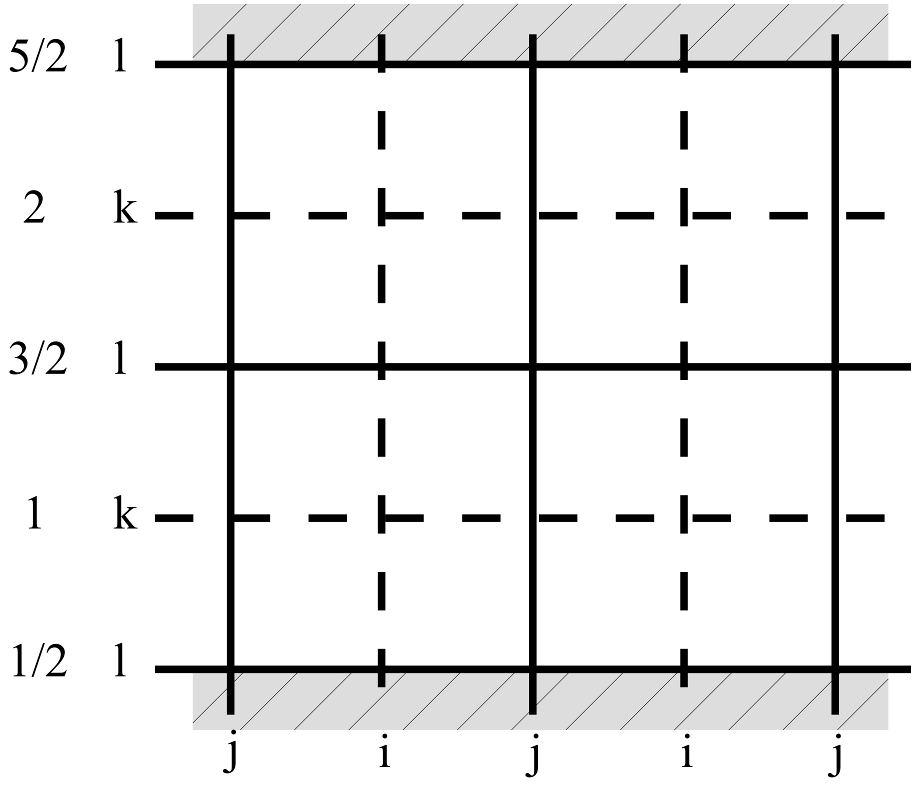
5 fields
Non-redundant flow description

$\mu,\ s \text{ or } S = \mu s,\ \Phi$
+ momentum
Lagrangian

$\mu,\ \Phi$
+ momentum
Isentropic / Isopycnal

$\mu = \int M \mathrm{d}\eta,\ s \text{ or } S = \mu s,\ \Phi$
+ momentum
Mass-based

$\mu,\ s \text{ or } S = \mu s$
+ momentum
Eulerian (z-based)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2 k
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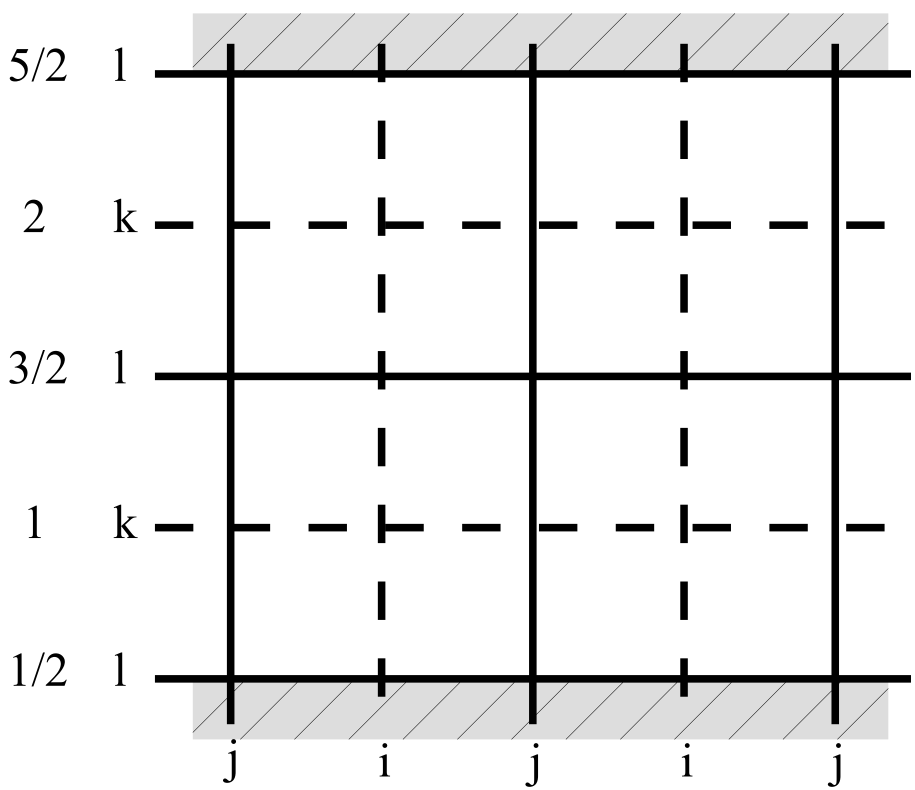

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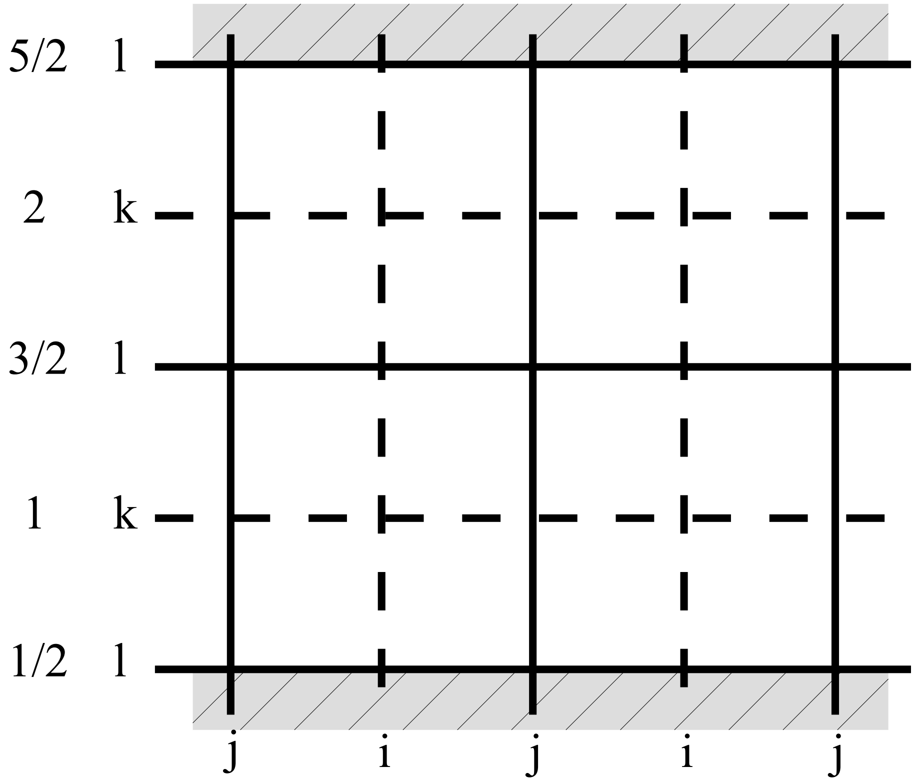

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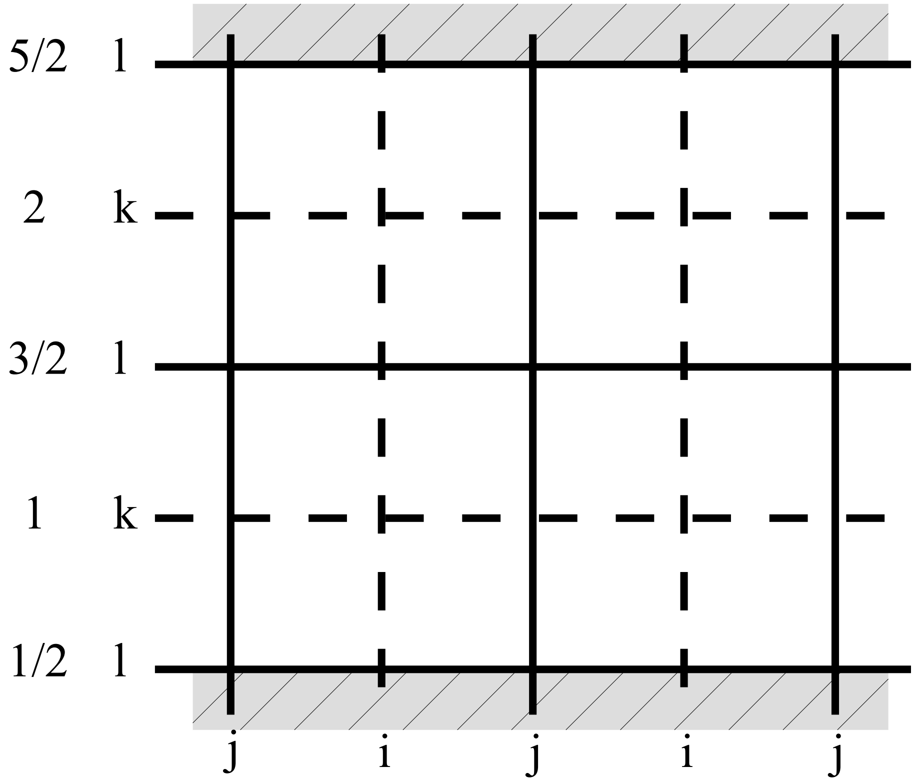

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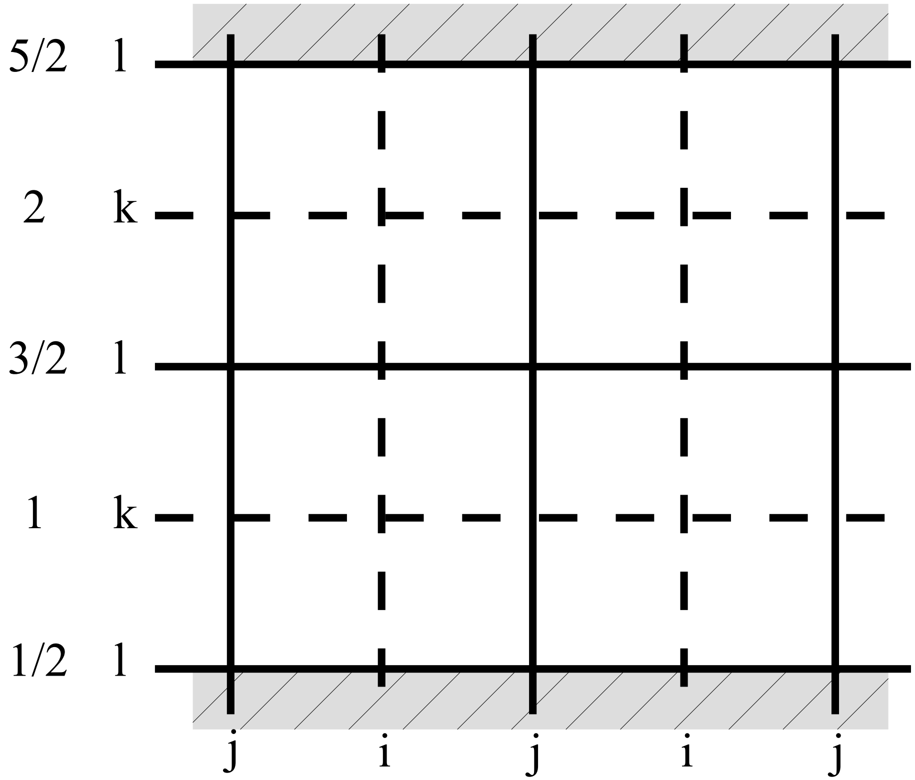

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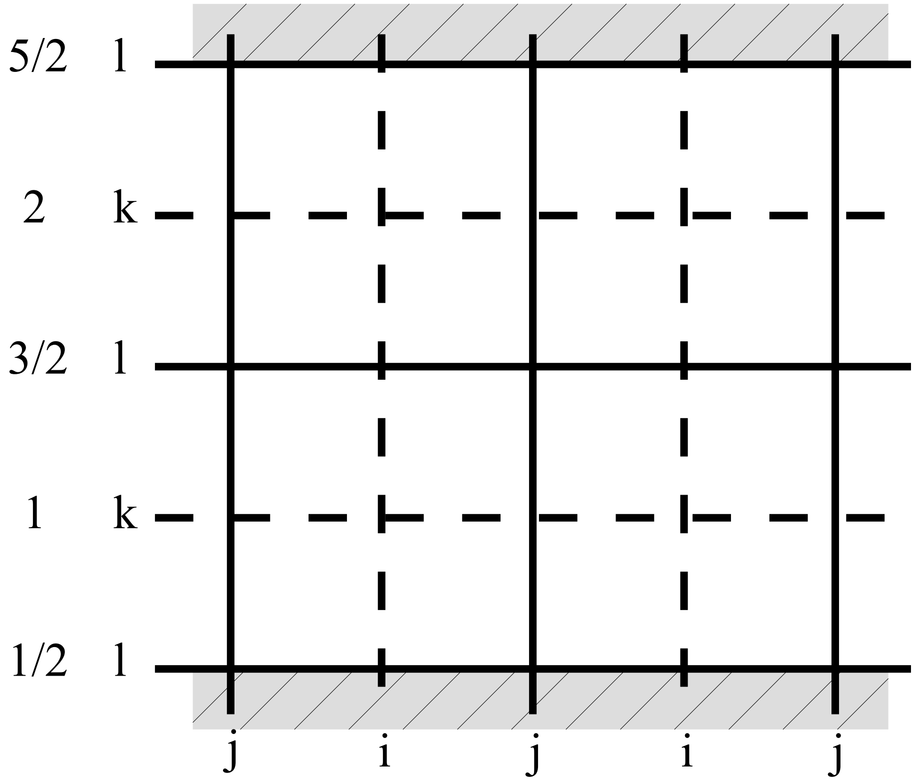

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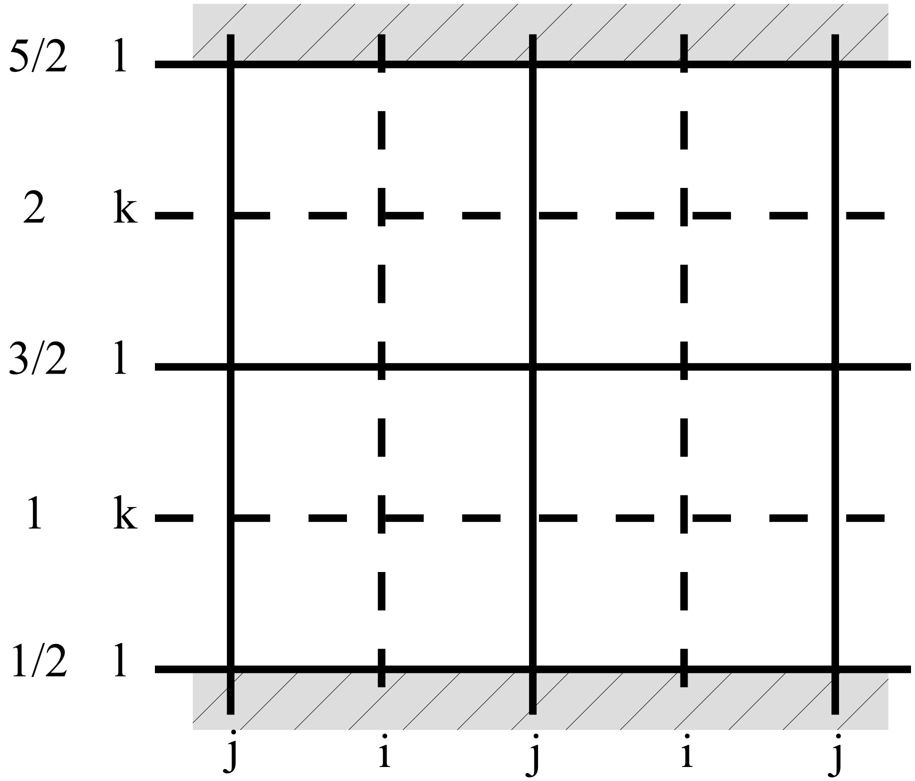

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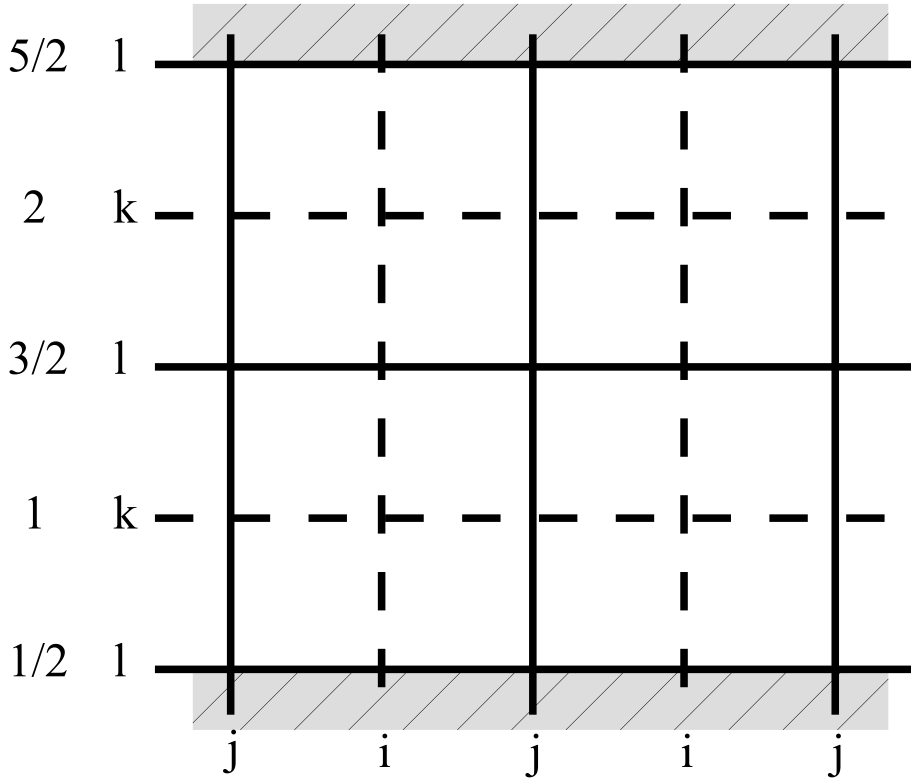

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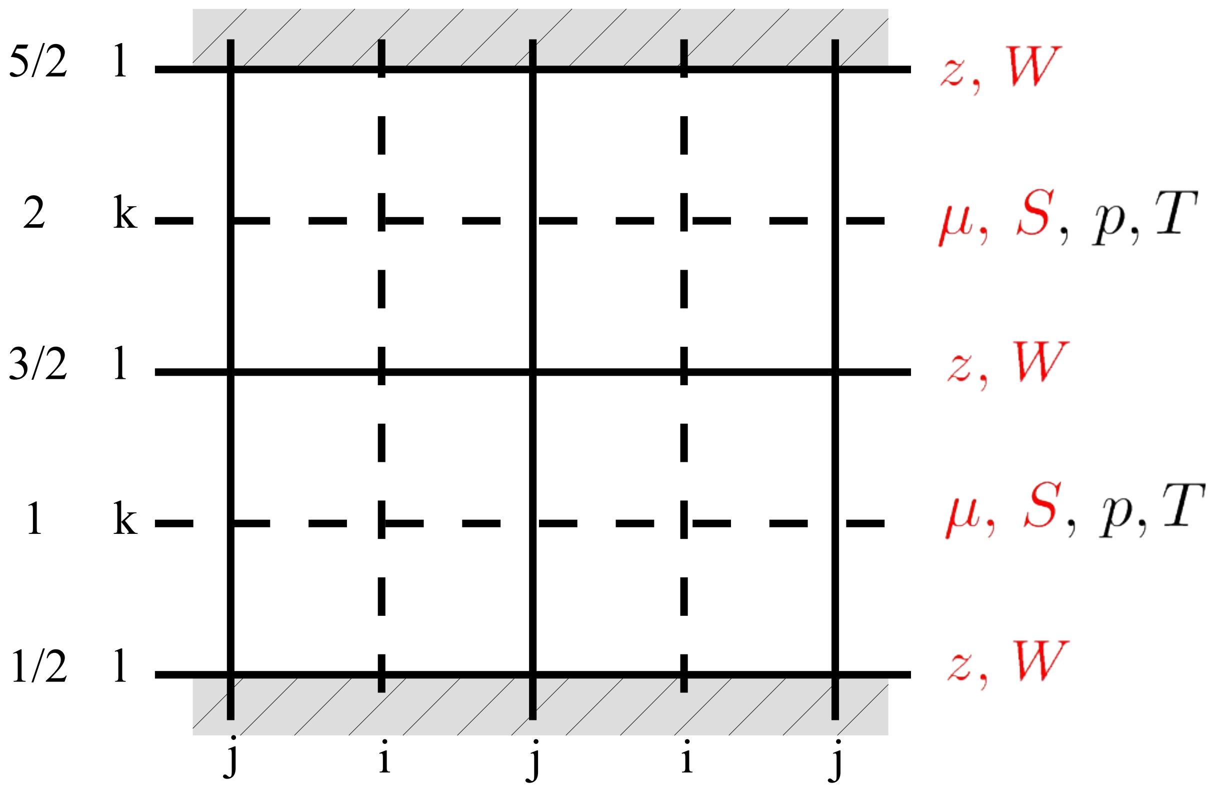


Height coordinate	Isentropic coordinate	Terrain-following mass-based coordinate
Category 1		
$(w\theta, uvp)$	$(wz, uvM)^M$ $(w, uv\sigma M)^{M\,a}$	$(w\theta, uvp)$
Category 2a (slow Rossby modes)		
$(w\theta, uv\rho)$ $(wT, uvp)^b$	(wz, uvp) $(wp, uvM)^{M\,b}$	$(w\theta z, uv)$ $(w\theta, uv\rho)$
Category 2b (fast Rossby modes)		
$(w\theta, uvT)$ $(w\rho, uvp)^b$	$(wz, uv\sigma)^p$	$(w\theta, uvT)^u$
Category 3		
$(w, uvp\rho)$ $(w, uvT\rho)$ $(w, uvp\theta)$ $(w, uv\theta\rho)$ $(w, uvpT)$ $(w, uv\theta T)$	$(w, uvpz)$ $(w, uvp\sigma)$ $(w, uvpM)$ $(w, uvzM)$	(wz, uvp) $(w, uvpT)$ (wz, uvT) $(w, uvT\rho)^u$ $(wz, uv\theta)$ $(w, uv\theta\rho)^u$ $(w, uvp\rho)^u$ $(w, uv\theta T)$ $(w, uvp\theta)$
Category 4		
$(wuv\theta, p)$	$(wuvz, M)^M$	$(wuv\theta, p)$

Thuburn & Woolings (2005)

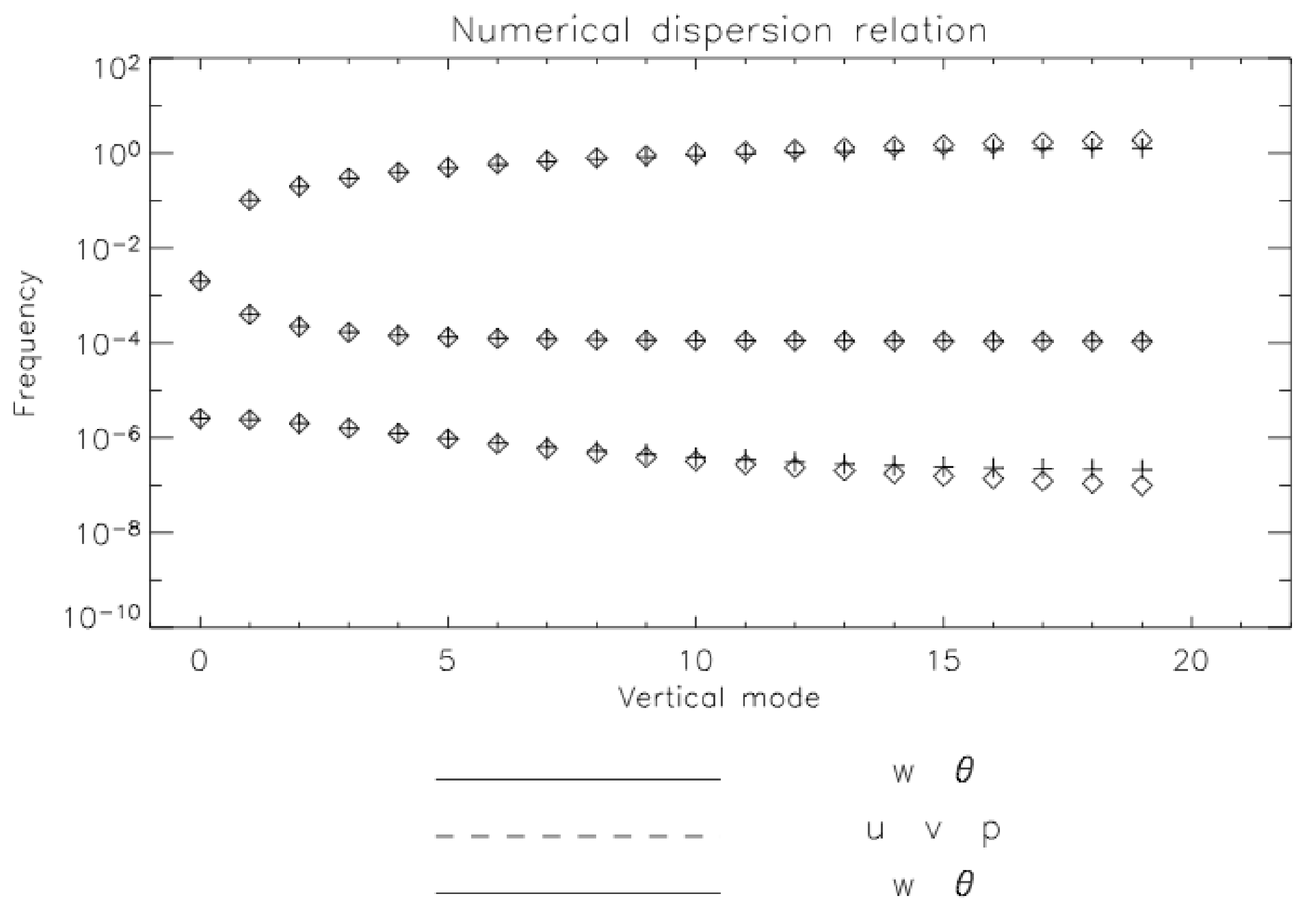


Fig. 1. Numerical dispersion relation (frequency in s^{-1}) for the optimal height-coordinate configuration ($w\theta, uvp$). The arrangement of variables on the grid is shown by the schematic underneath the main graph. Crosses indicate frequencies of numerical eigenmodes; diamonds indicate frequencies of analytical eigenmodes. Only westward propagating modes are shown; the behaviour of the eastward propagating acoustic and inertia-gravity modes is extremely similar to that of their westward-propagating counterparts.

Thuburn & Woolings (2005)

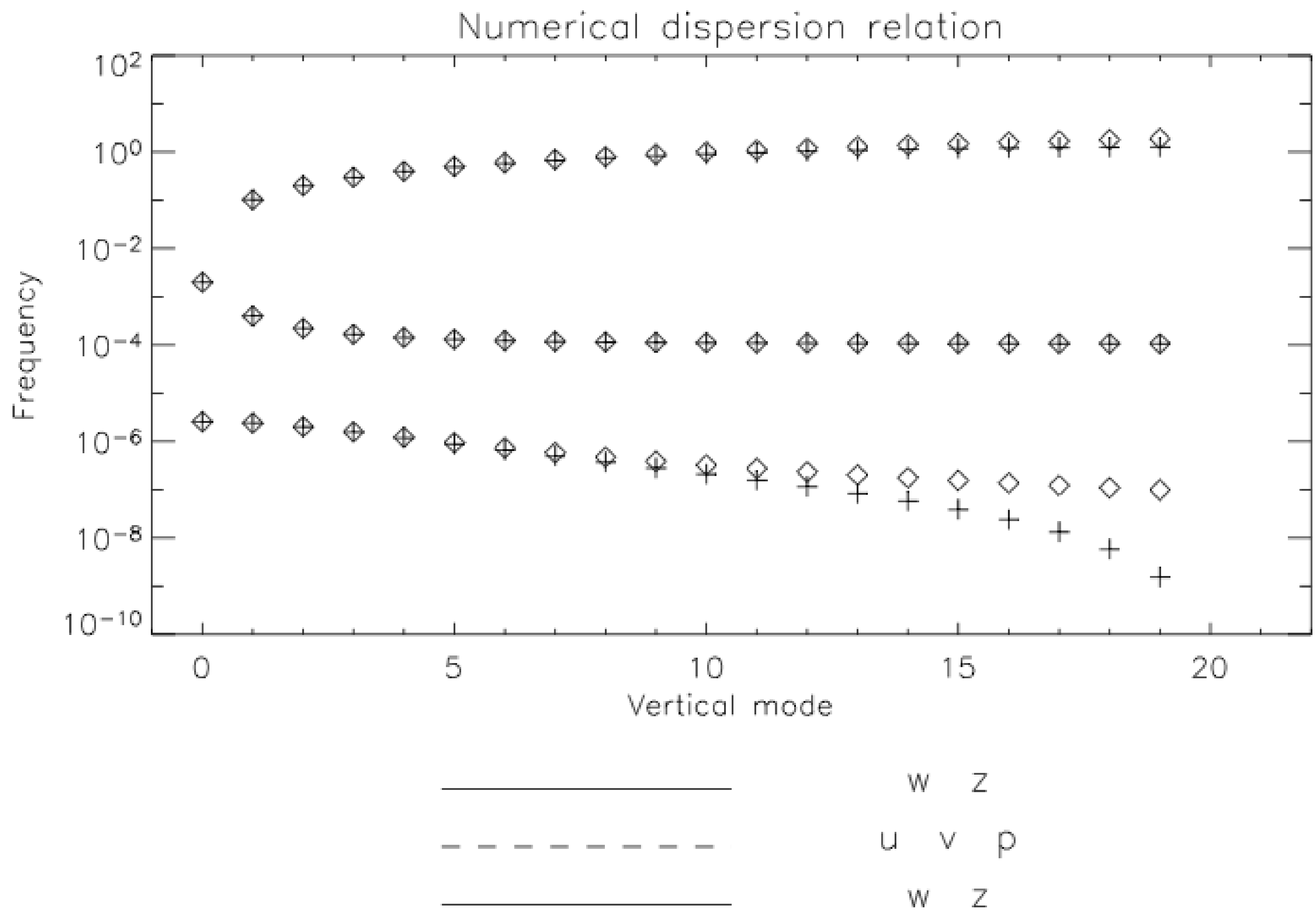


Fig. 2. Numerical dispersion relation for the category 2a isentropic-coordinate configuration (wz, uvp). Other details as in fig. 1.

Thuburn & Woolings (2005)

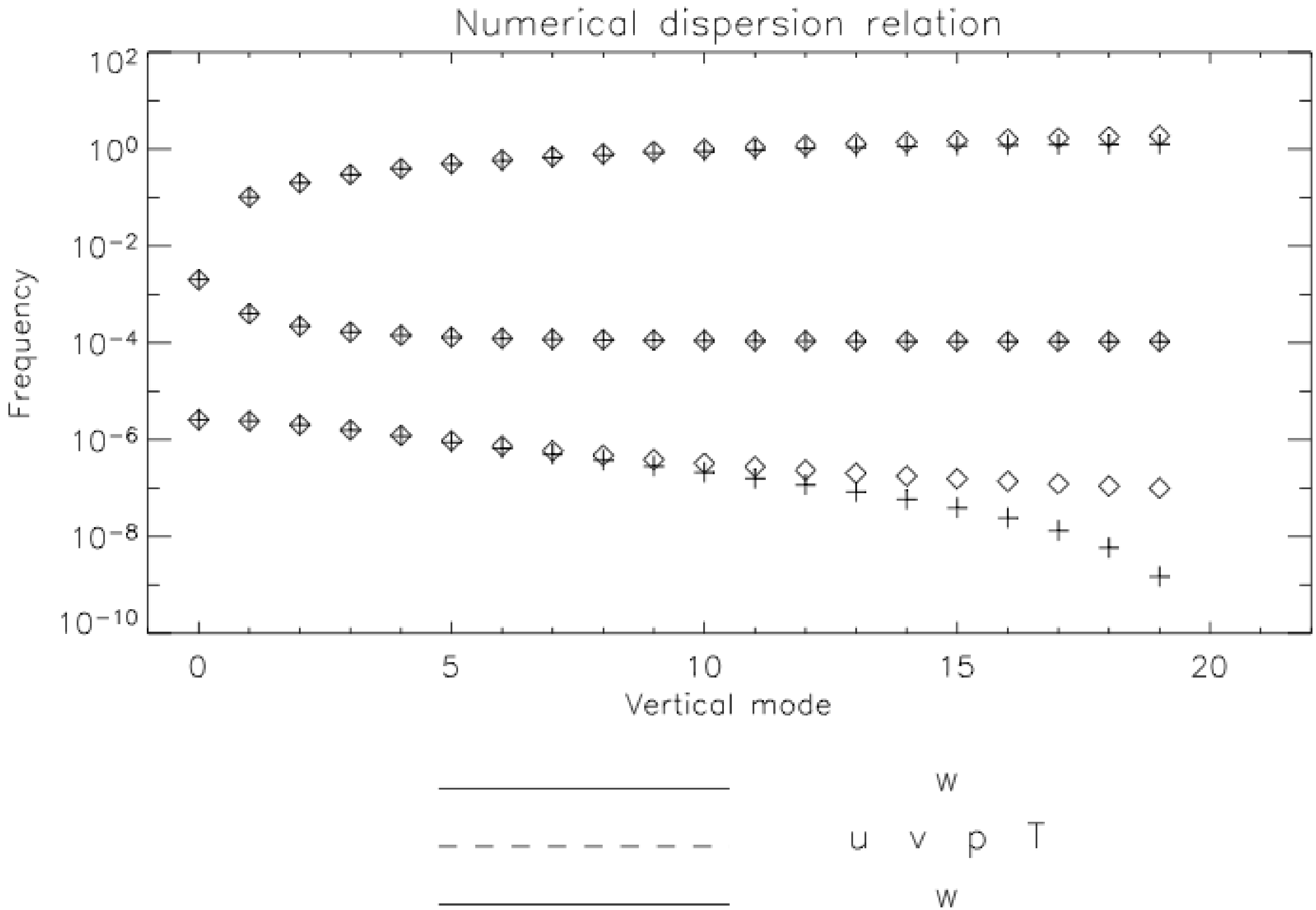
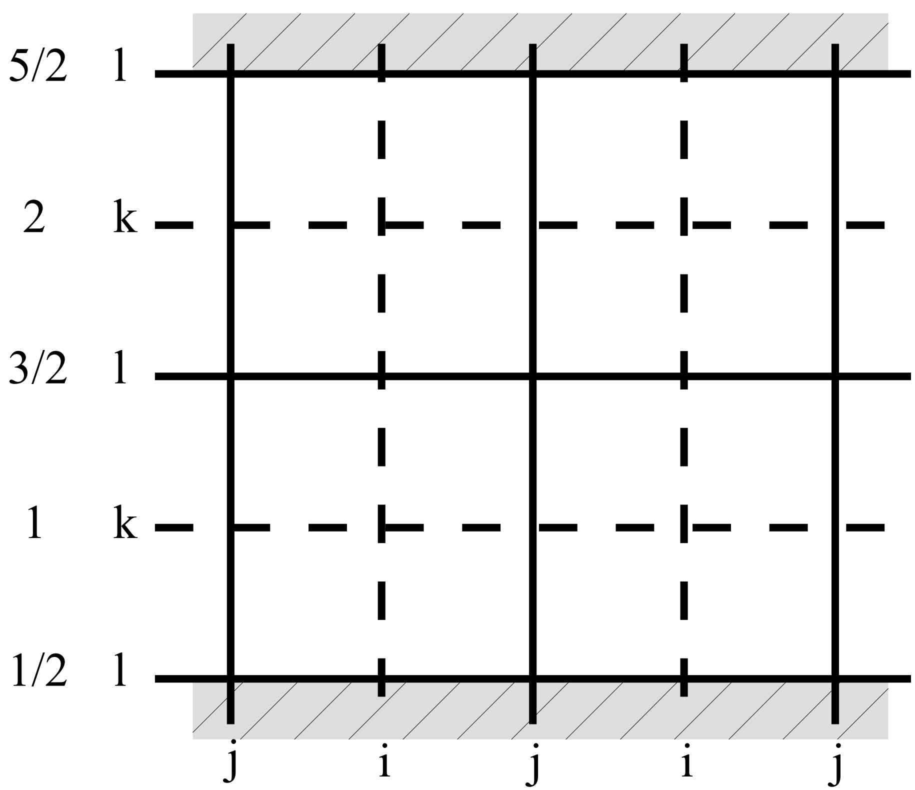


Fig. 4. Numerical dispersion relation for the category 3 terrain-following mass-based coordinate configuration $(w, uvpT)$. Other details as in fig. 1.

Category 3: Single zero-frequency computational mode

Thuburn & Woolings (2005)



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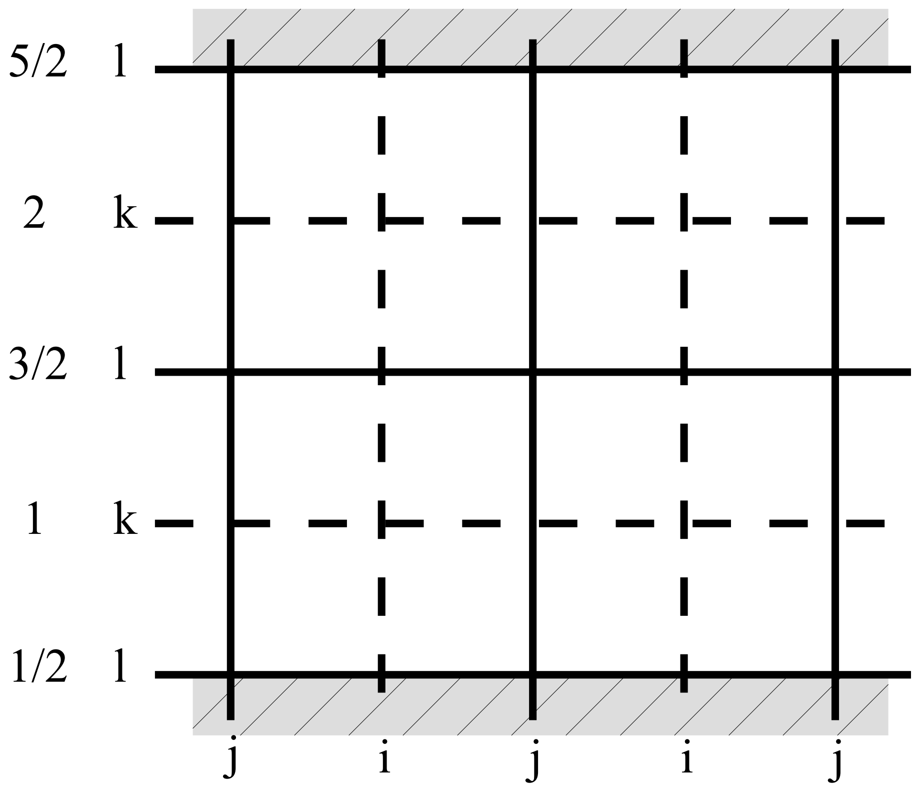

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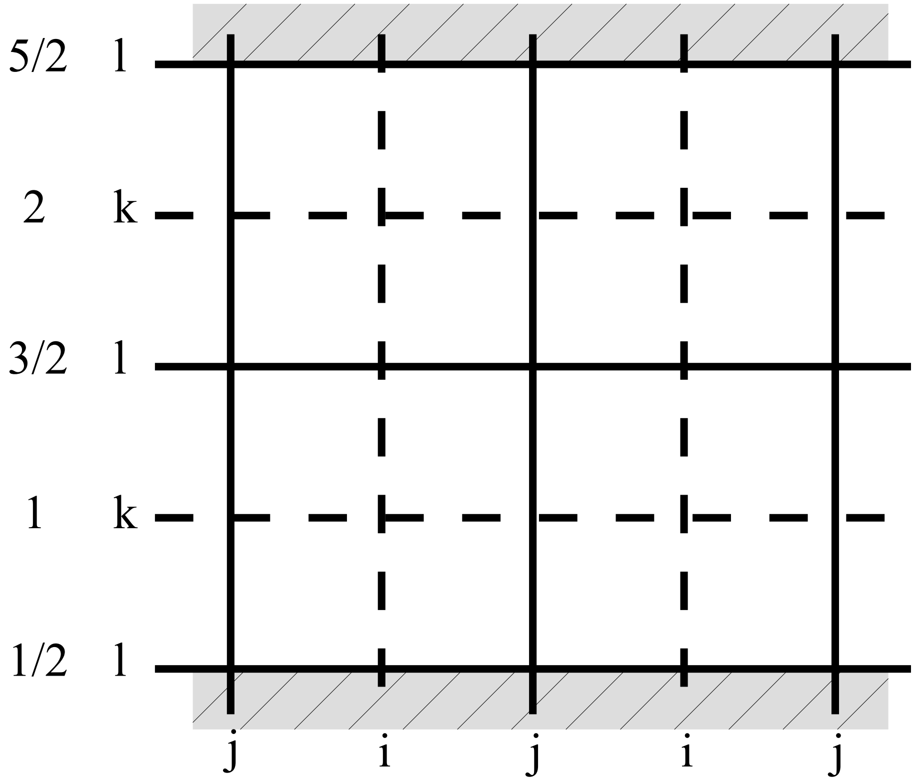

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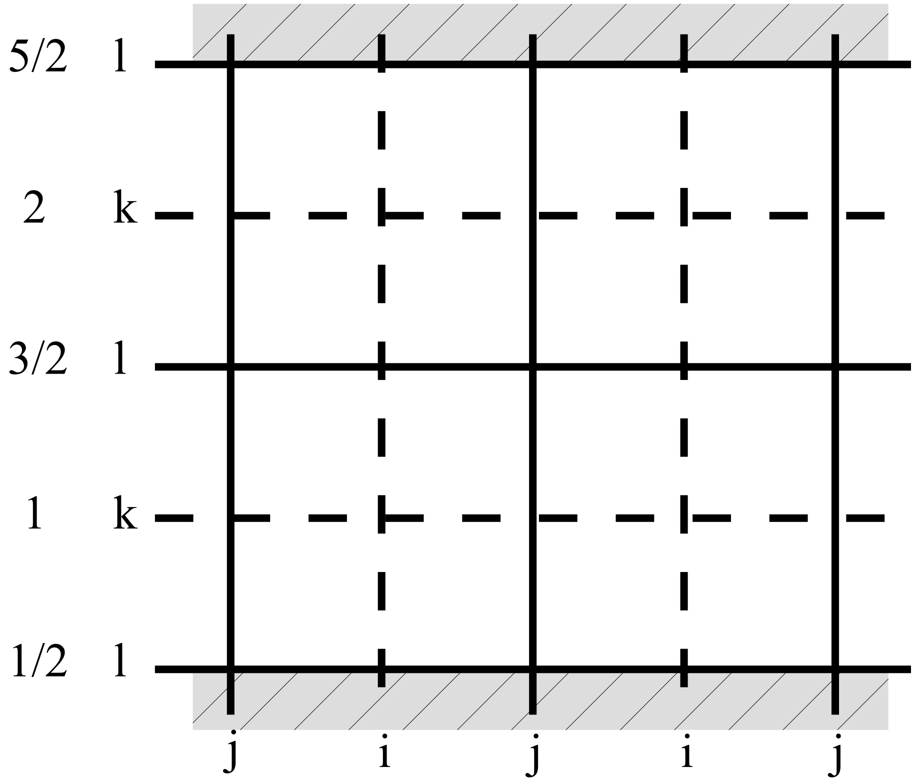

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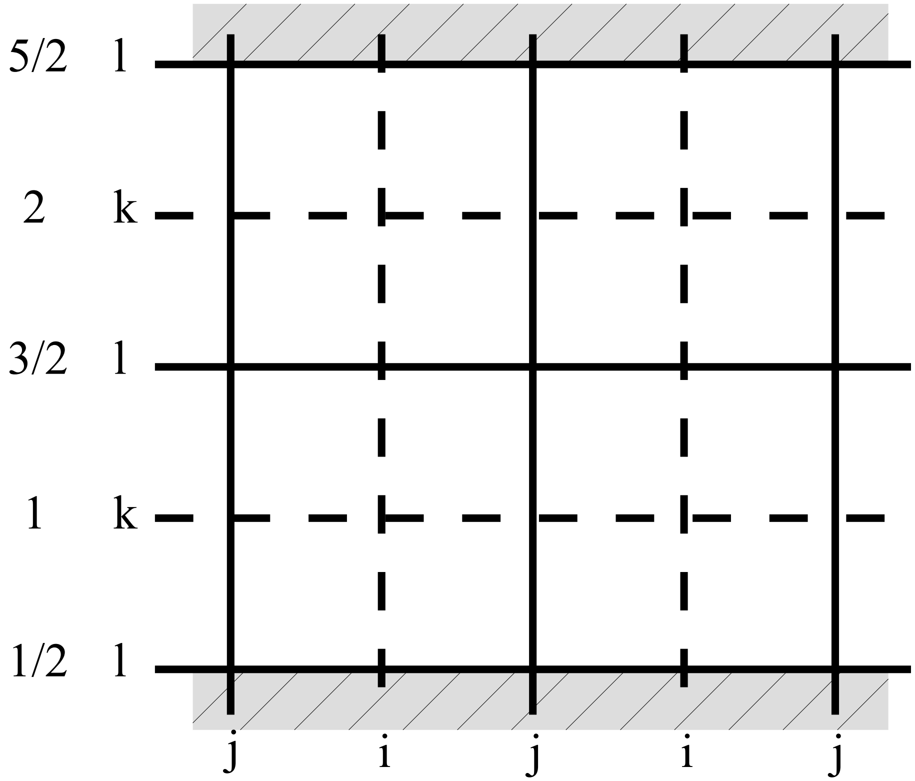

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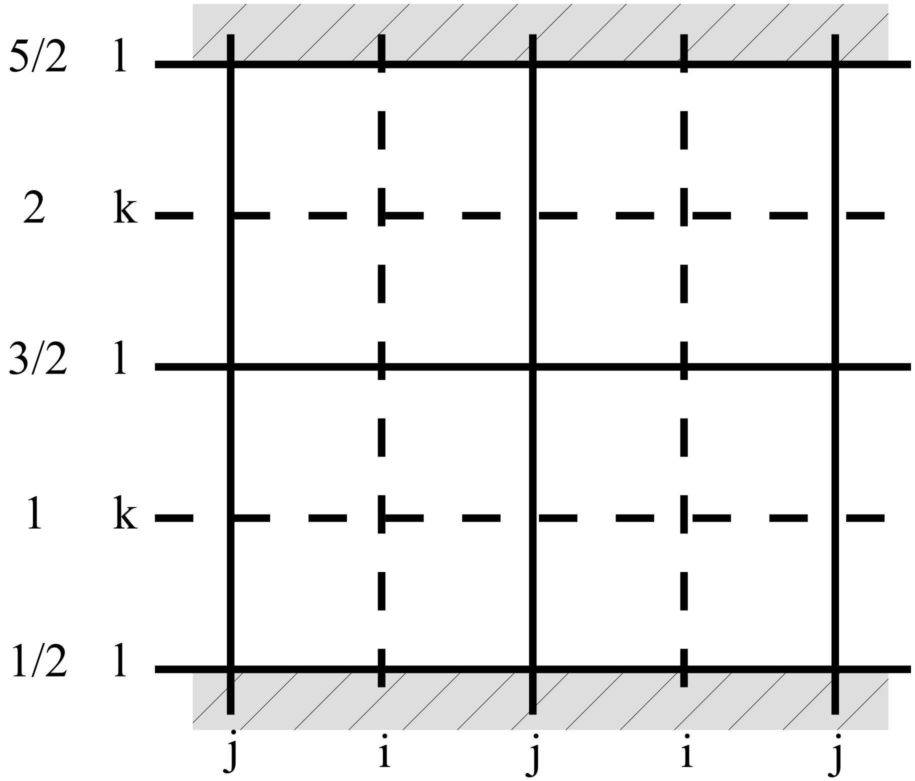

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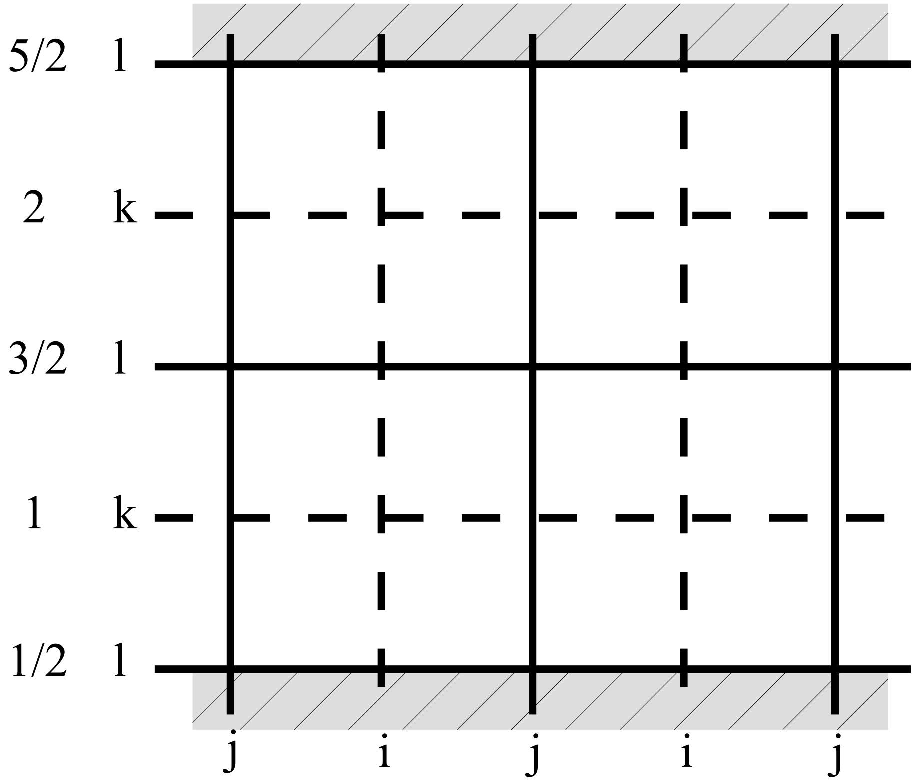

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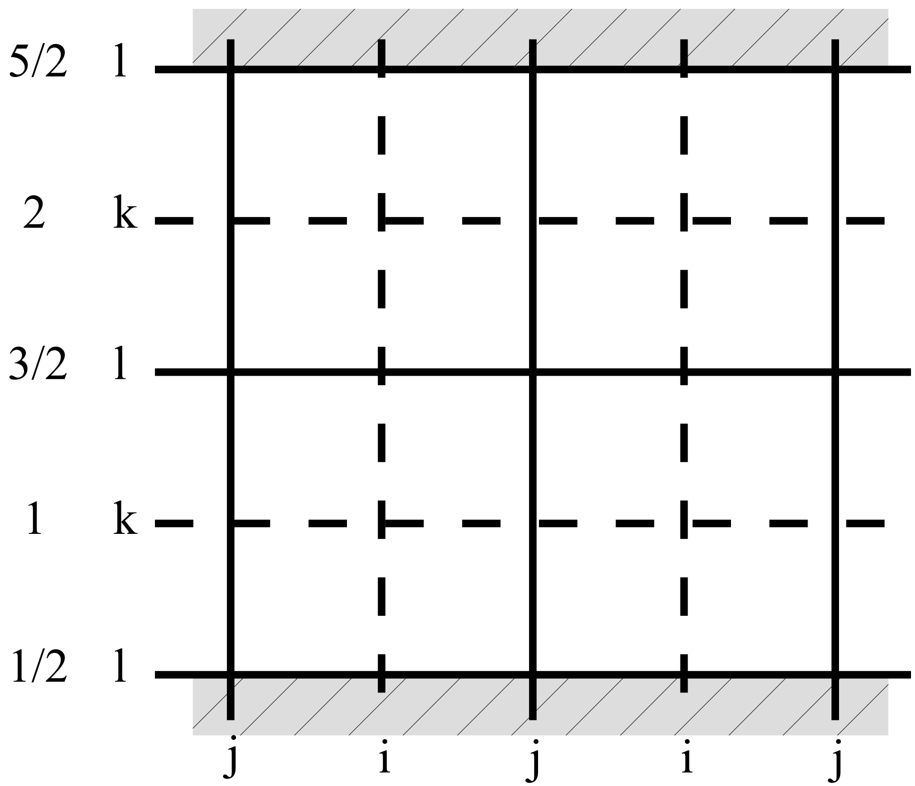

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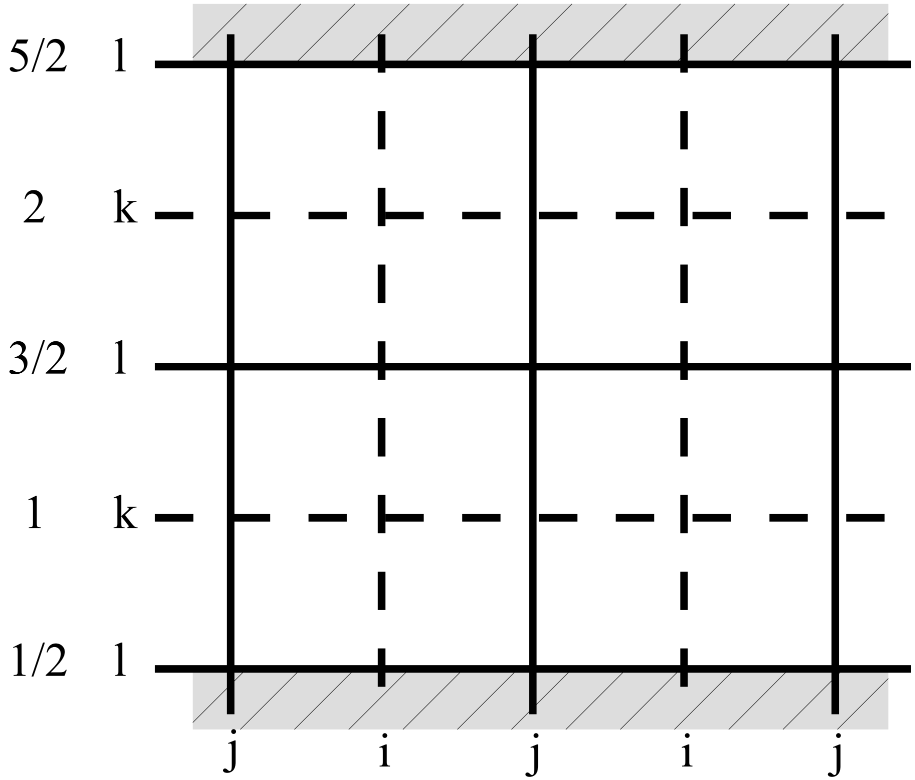

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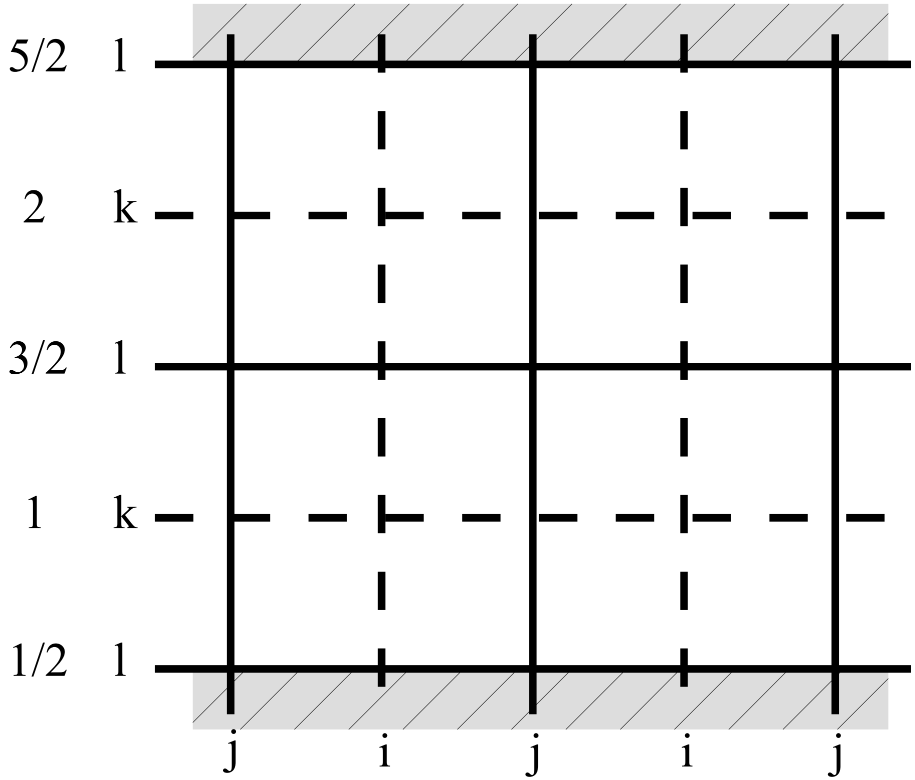

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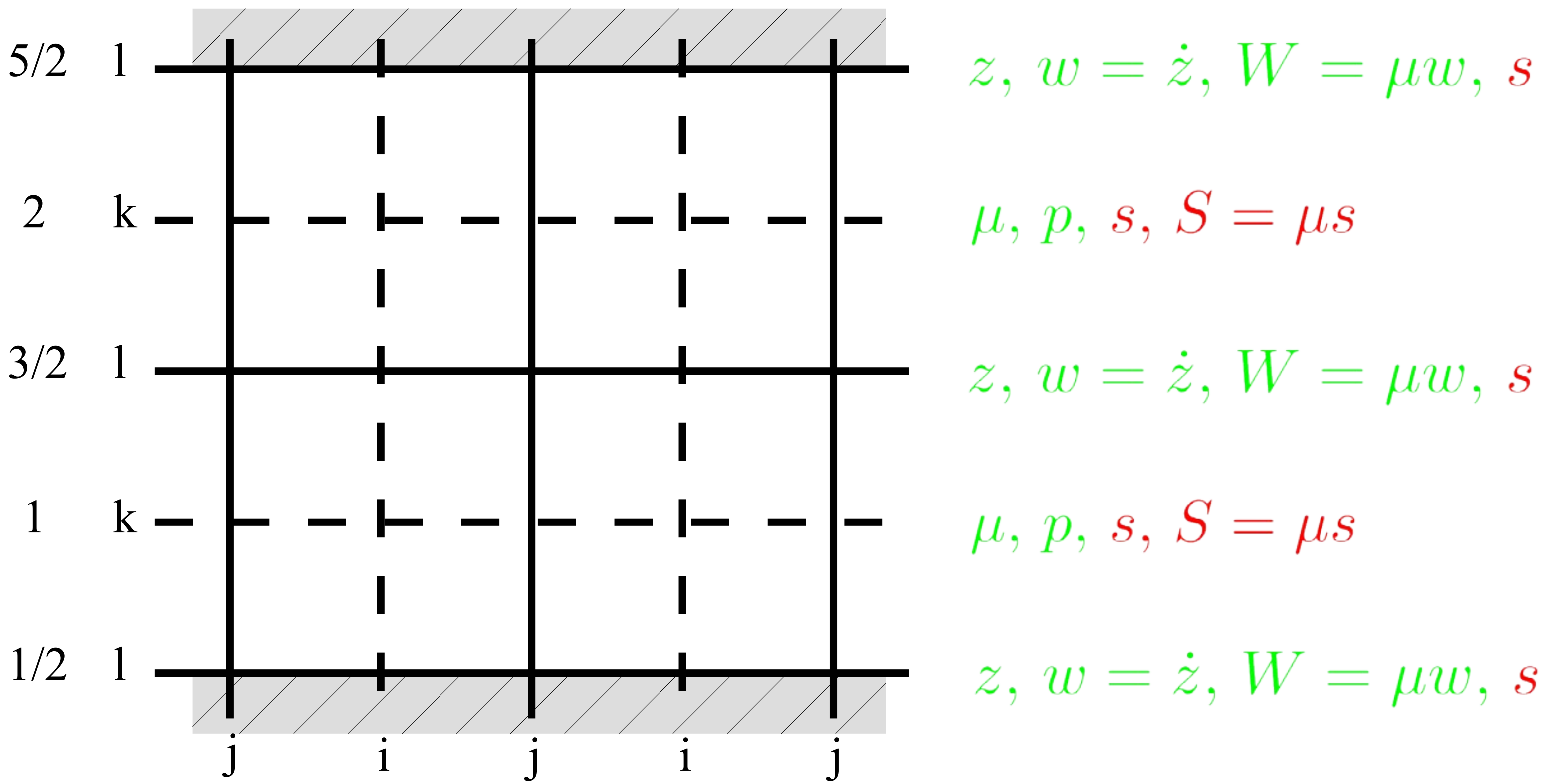
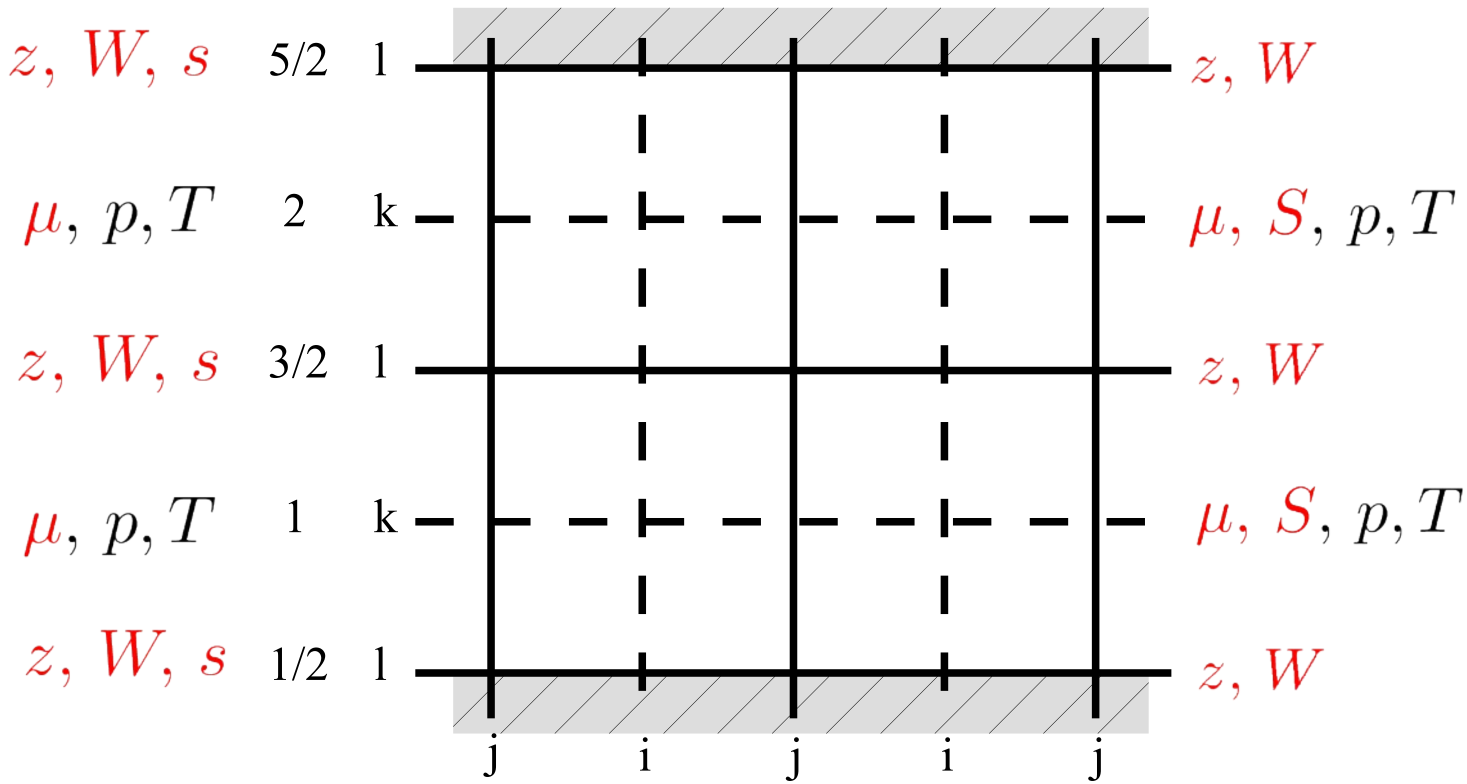


Charney-Phillips staggering **Lorenz staggering**



Some atmosphere models
MetOffice

Most atmosphere / ocean models
LMDZ, WRF, NEMO

Vertical discretization :

- Vertical coordinates
- Where should we place prognostic/diagnostic variables
- Criteria : mass/transport consistency, numerical dispersion

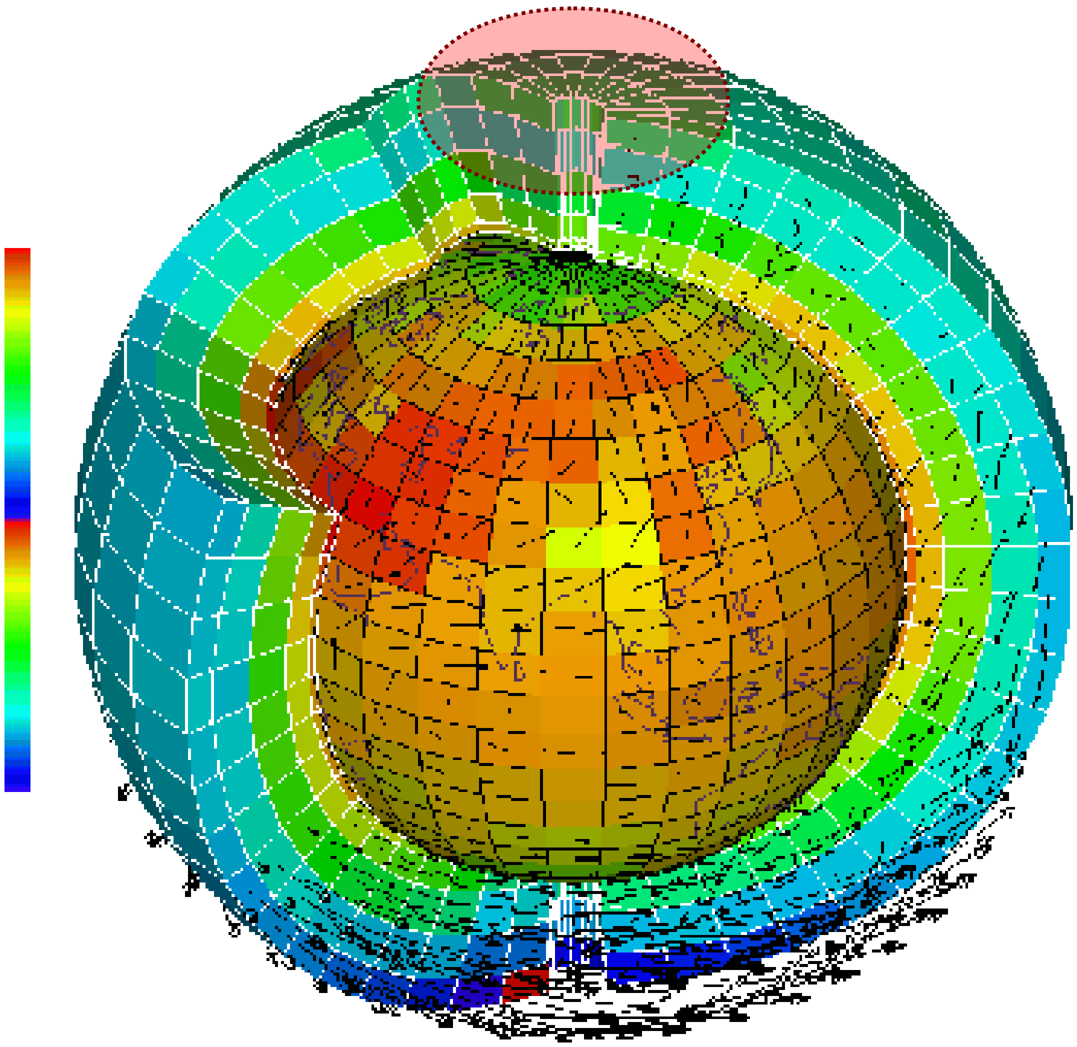
Spherical meshes

- **curvilinear Cartesian meshes**
- **global curvilinear Cartesian meshes : the pole problem**
- **a meshless method : spectral method**
- **quasi-uniform meshes and associated issues**

Conservation of non-linear integral invariants

- a few clever solutions
- towards systematic approaches

The pole : problem and solutions



Regular 2NxN lon-lat mesh

$$\delta\lambda = \pi / N$$ cell size (rad)

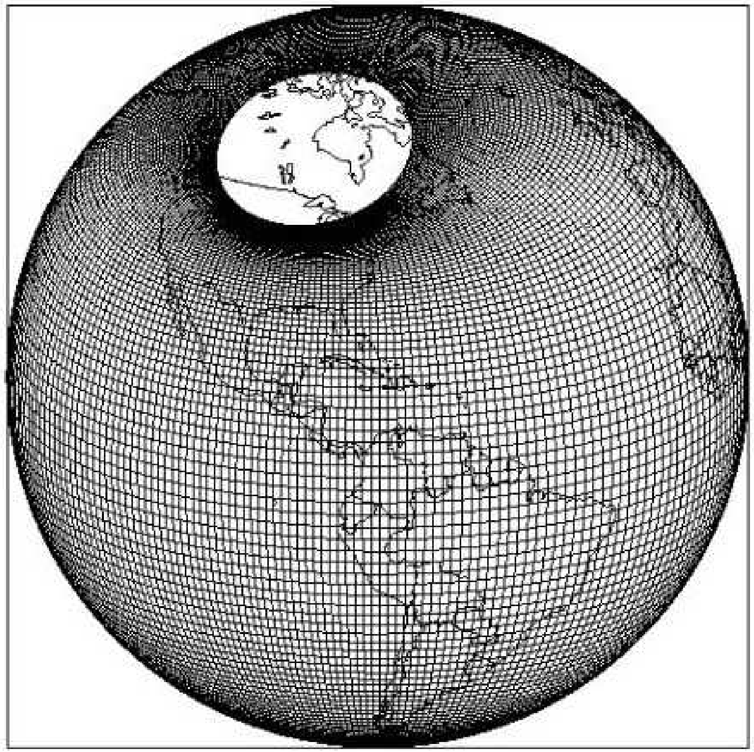
$$a \cos\phi \delta\lambda = a\pi^2 / N^2$$ cell size (m)

$$\delta t \sim \frac{a}{cN^2}$$

Solutions

- Spectral method
- Zonal filters
- Quasi-uniform meshes

Curvilinear Cartesian meshes



- Any curvilinear Cartesian mesh covering the sphere must have singularities
- For ocean modelling they can be placed on continents
- Still quite non-uniform, but acceptable
- For atmosphere : cannot remove the singulatities

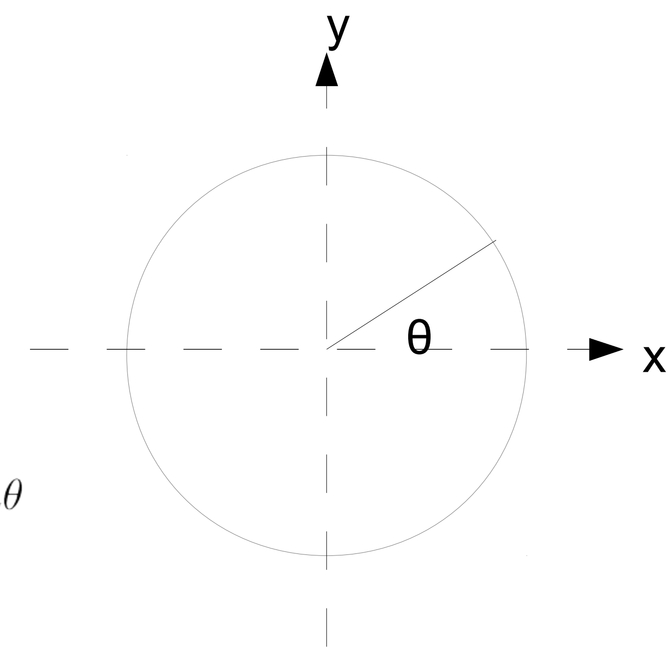
Spherical harmonics
an analogy wih 1D Fourier decomposition

- 1D periodic functions = functions on the unit circle
- Fourier series = polynomials on the circle $x^2+y^2=1$

$$e^{in\theta} = (x+iy)^n$$

- *Sort* them according to *spatial scale* :
  eigenvalues of Laplacian
  associated eigenmodes = Fourier modes
- *Project, expand* and *truncate* at |n|<N

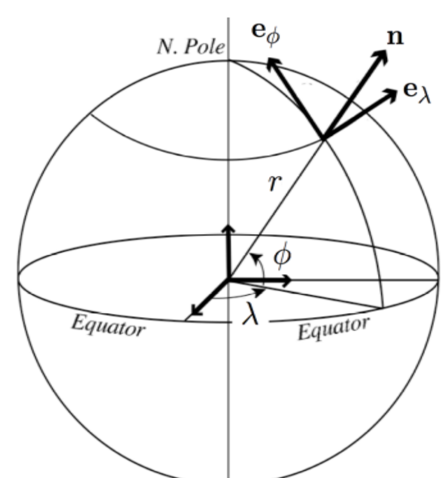
$$\hat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \qquad f(\theta) = \sum_{n=-N}^{N} \hat{f}_n \exp in\theta$$

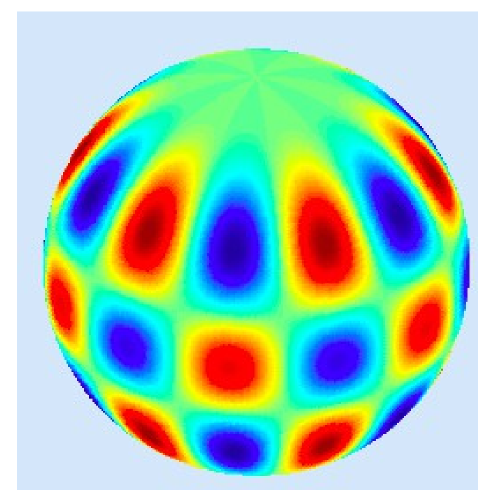


- Smooth functions on the unit sphere $x^2+y^2+z^2=1$
  = polynomials in Cartesian coordinates x,y,z

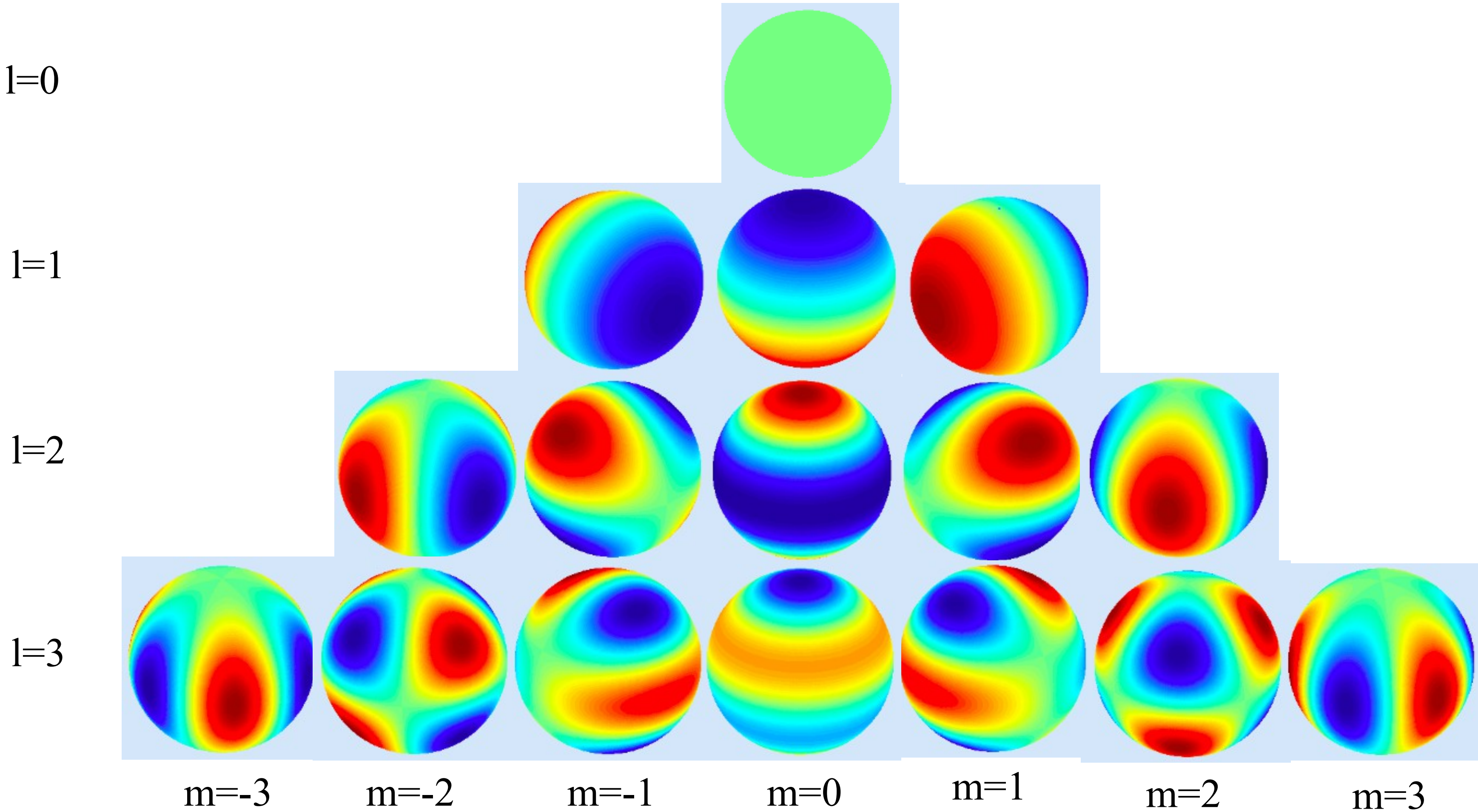
$$\begin{aligned} x &= \cos\phi\cos\lambda \\ y &= \cos\phi\sin\lambda \\ z &= \sin\phi \end{aligned} \qquad f_{ijk}(\lambda,\phi) = x^i y^j z^k$$

- Sort them according to spatial scale :
  eigenvalues of Laplacian
  associated eigenmodes = spherical harmonics





$$\Delta f = -l(l+1)f \qquad Y_{lm} = P_{lm}(\sin\phi)\exp im\lambda$$



Although the *basis* of each eigenspace is anisotropic, each eigenspace *is isotropic*
=> *good basis for uniform-resolution representation of scalar fields using triangular truncation*

$$\hat{f}_{lm} = \frac{1}{4\pi}\int Y_{lm}(\lambda,\phi) f(\lambda,\phi)\cos\phi \, d\lambda d\phi \qquad f(\lambda,\phi) = \sum_{l=0}^{l=L}\sum_{m=-l}^{m=l}\hat{f}_{lm}Y_{lm}(\lambda,\phi)$$

- Representation of vector fields : vorticity-streamfunction decomposition
- Very accurate if the fields are very smooth : not really relevant for atmosphere/ocean

Harmonic transform in practice

Forward transform

- Integrals are computed as weighted sums of pointwise values (*quadrature formula*)
- Zonal : regularly spaced, equal weights => 1D FFT
- Latitudinal : unequally spaced, Gauss-Legendre weights => L full matrix-vector multiplications lxl

$$\begin{aligned}
\hat{f}_m(\phi_j) &= \frac{1}{2\pi}\int e^{-im\lambda} f(\lambda,\phi_j)\mathrm{d}\lambda \\
&\simeq \frac{1}{N}\sum_i \int e^{-im\lambda_i} f(\lambda_i,\phi_j) \\
\hat{f}_{lm} &= \frac{1}{4\pi}\int P_{lm}(\phi)\hat{f}_m(\phi)\cos\phi\,\mathrm{d}\lambda\mathrm{d}\phi \\
&\simeq \sum_j \hat{f}_m(\phi_j)P_{lm}(\phi_j)w_j
\end{aligned}$$

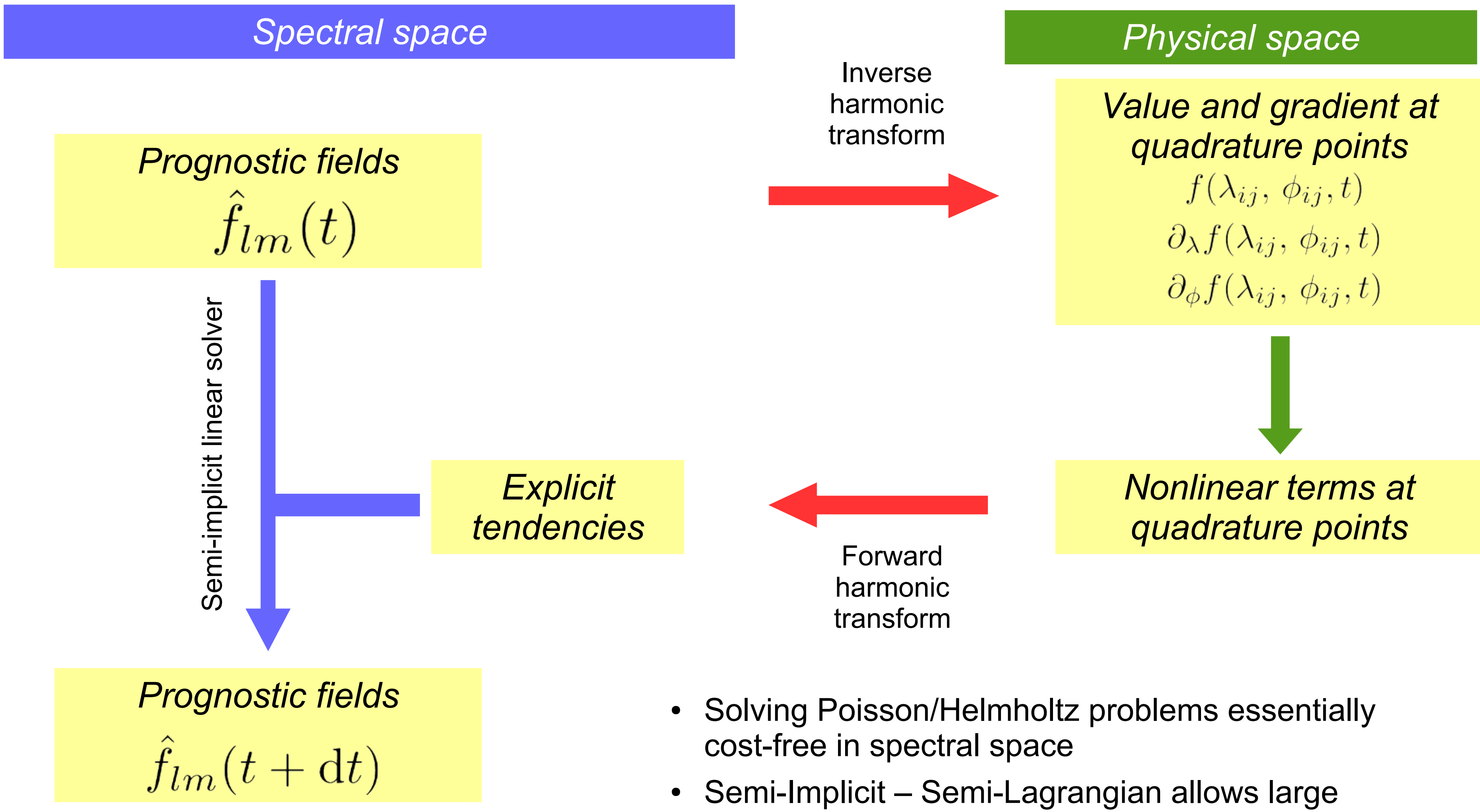
Backward transform

- Latitudinal : L full matrix-vector multiplications
- Zonal : 1D FFT

Although the *quadrature points* form a non-uniform latitude-longitude mesh, the resolution *is* uniform

- One needs usually Nx=3L and Ny=3L/2 points to avoid *aliasing of nonlinear terms* : about 4 quadrature points for one spectral coefficient
- Cost of FFT reasonable O(Ny.Nx.log(Nx)) but hard to parallelize efficiently
- Matrix-vector multiplications possible to parallelize but expensive O(L.Ny^2)
- Imminent death of the spectral method predicted regularly in the last 30 years
- Still there after huge efforts for efficiency at ECMWF (Wedi, 2013)

Typical sequence of operations in a spectral model



- Solving Poisson/Helmholtz problems essentially cost-free in spectral space
- Semi-Implicit – Semi-Lagrangian allows large time steps
- Compensates high cost of harmonics transforms
- Semi-Lagrangian not conservative

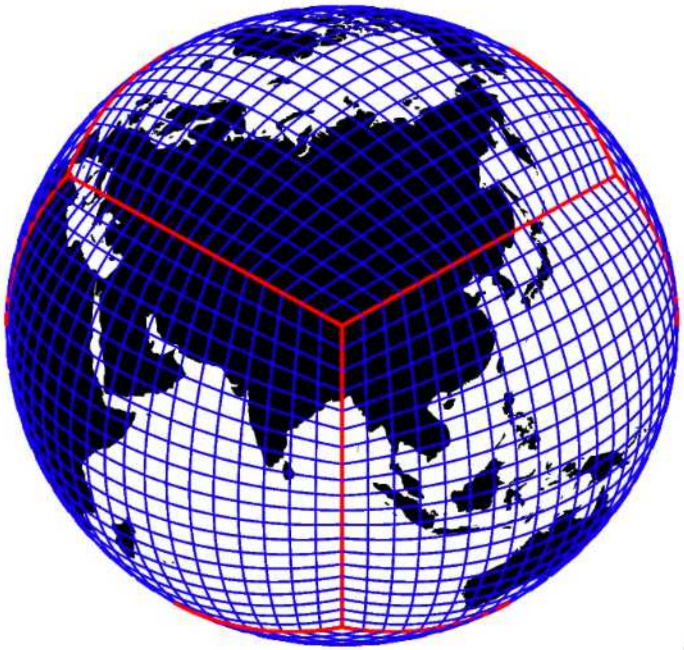
Spherical harmonics / spectral models : recap

- Spherical harmonics solve the pole problem by providing uniform-resolution function spaces
- Elliptic problems with horizontally uniform coefficients efficiently solved in spectral space
- Harmonic transforms expensive and hard to parallelize because spherical harmonics have global support
- Spectral semi-implicit semi-lagrangian (SISL) still the « method to beat » for numerical weather prediction

- Example of another approach to representing scalar/vector fields : expand/project on function bases (Galerkin approach)
- Active current research on Galerkin approach with locally supported basis functions (finite elements)

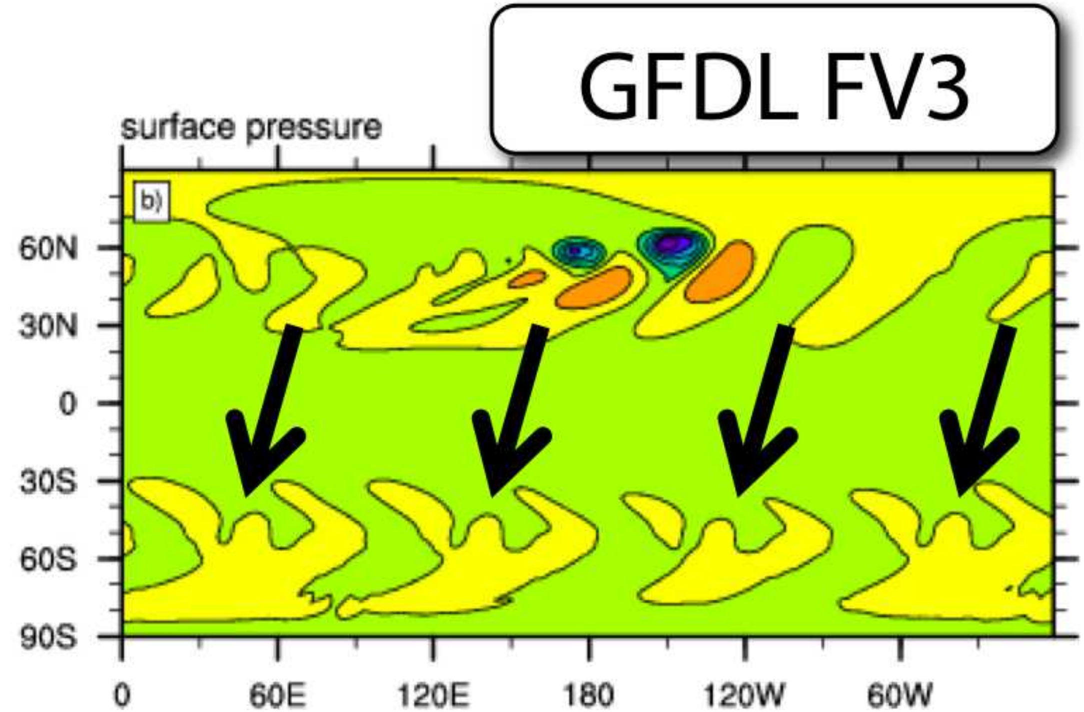
Quasi-uniform meshes

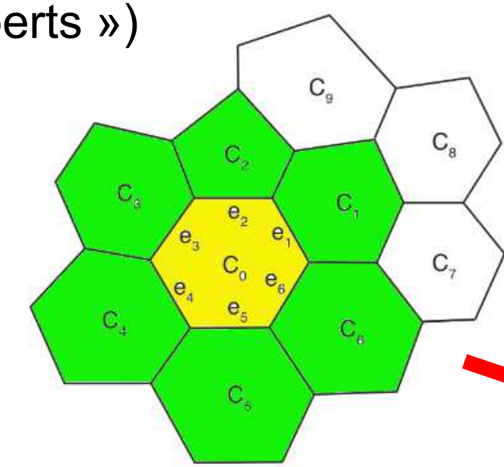
- Already considered in the early days of atmospheric modelling (*Sadourny et al., 1968*)
- Could not achieve satisfactory stability and conservation properties
- Abandoned in favor of spectral method or zonal filters
- Revived recently in search of more parallelism

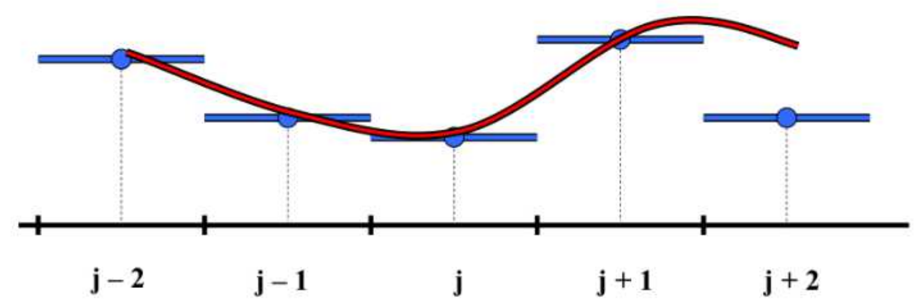


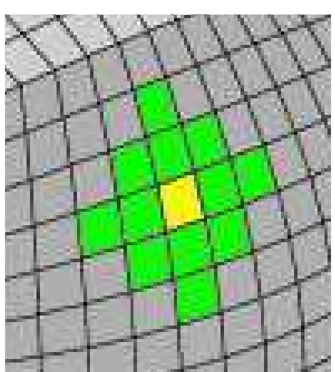
Stratégies pour l'ordre « élevé »

- Objectif : avant tout réduire l'empreinte de la grille
- Ordre 2 a priori suffisant, pourquoi pas ordre 3 ou 4 si efficace
- Plus important pour le transport que pour la dynamique (opinion des « experts »)



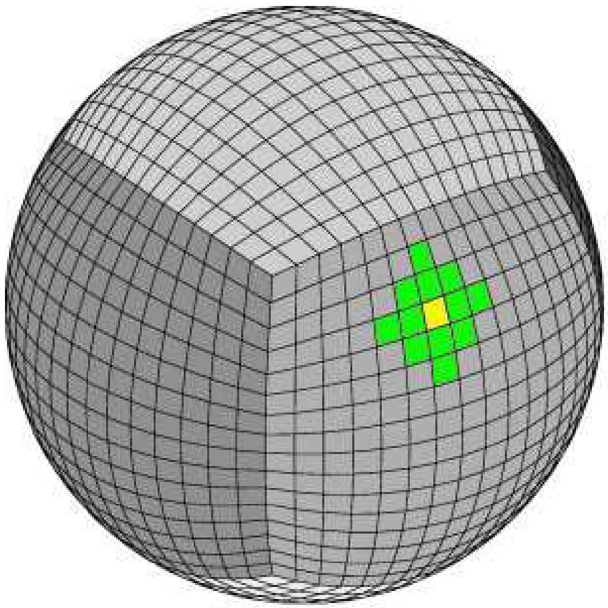


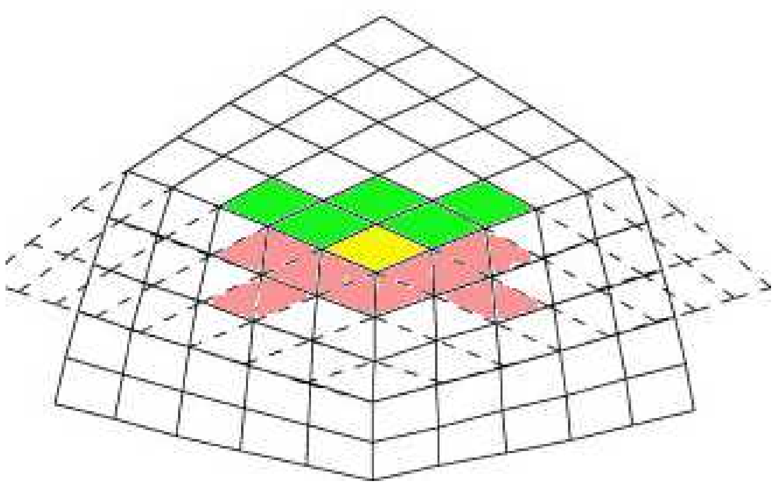


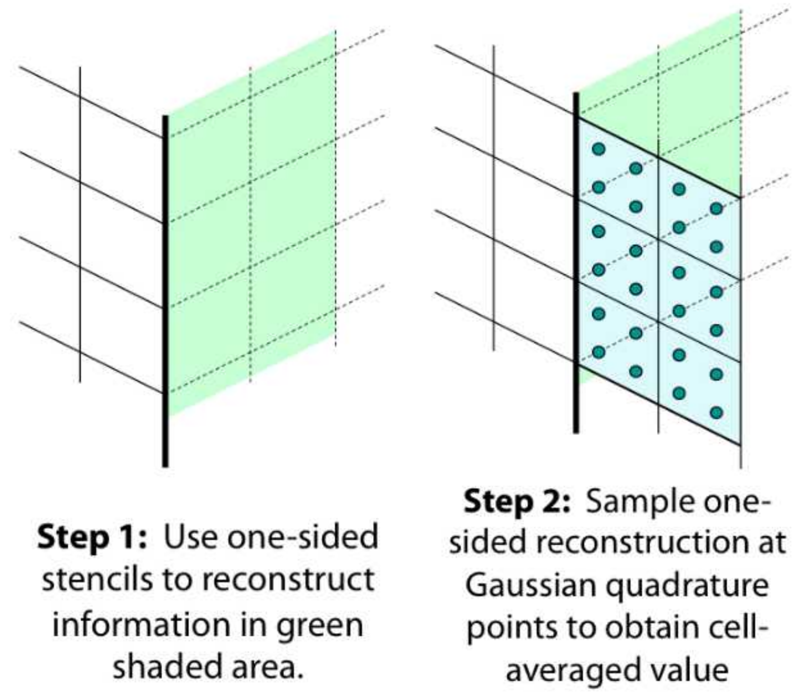


Stratégies pour l'ordre « élevé »

Ullrich, Jablonowski & Van Leer (2010) :
Volumes finis + reconstructions locales d'ordre 4 + Runge-Kutta semi-implicite



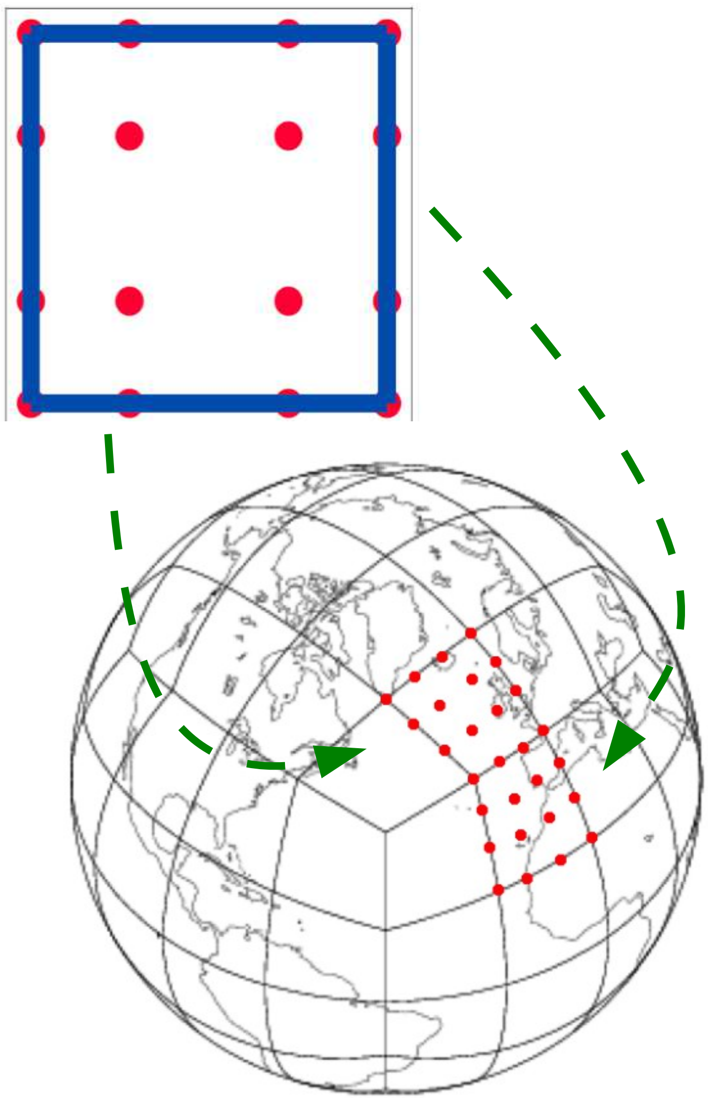




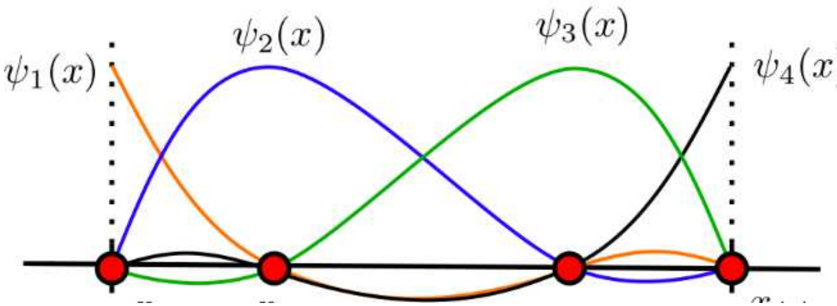
Stabilisation par dissipation implicite
(solveurs de Riemann)

Stratégies pour l'ordre « élevé »

Taylor & Fournier (2010) : Éléments finis d'ordre 4 (Q3) + Runge-Kutta,



$$q(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x)$$

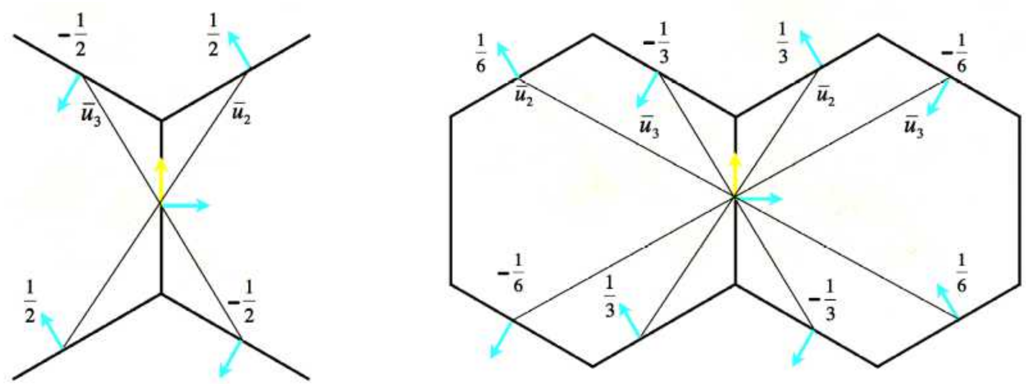


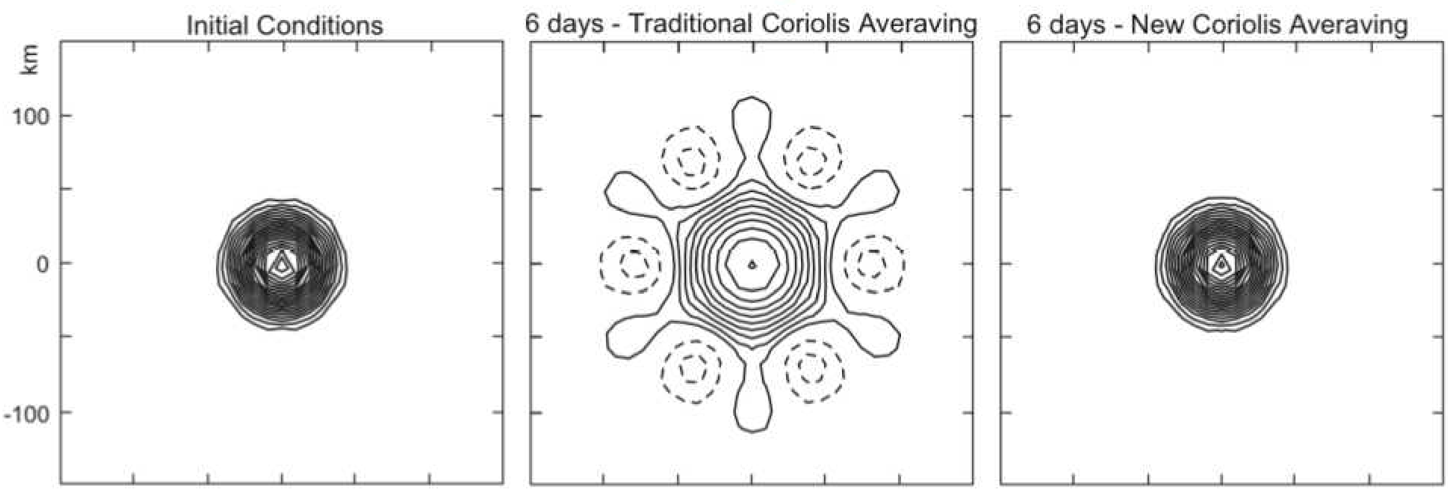
Quadrature aux points GLL
 => matrice de masse diagonale

Forme vector-invariant
+ compatibilité grad/div
=> conservation de l'énergie

Stabilisation par dissipation explicite
(hyperlaplacien)

Préservation discrète de l'équilibre géostrophique





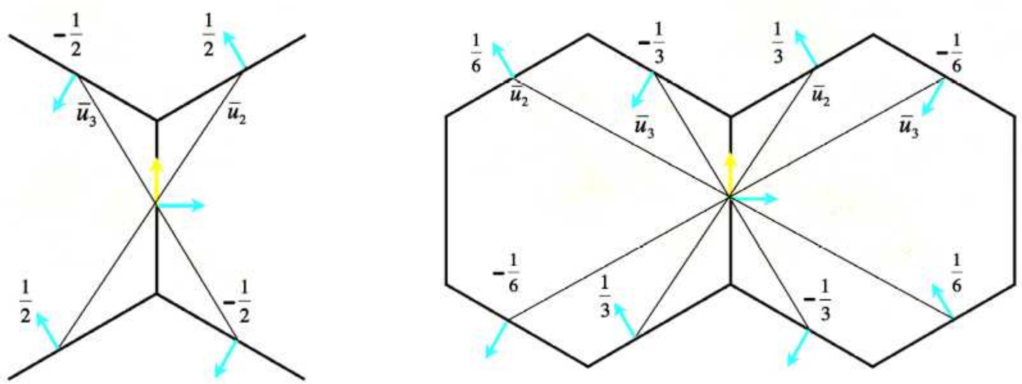
Équilibres géostrophiques violés

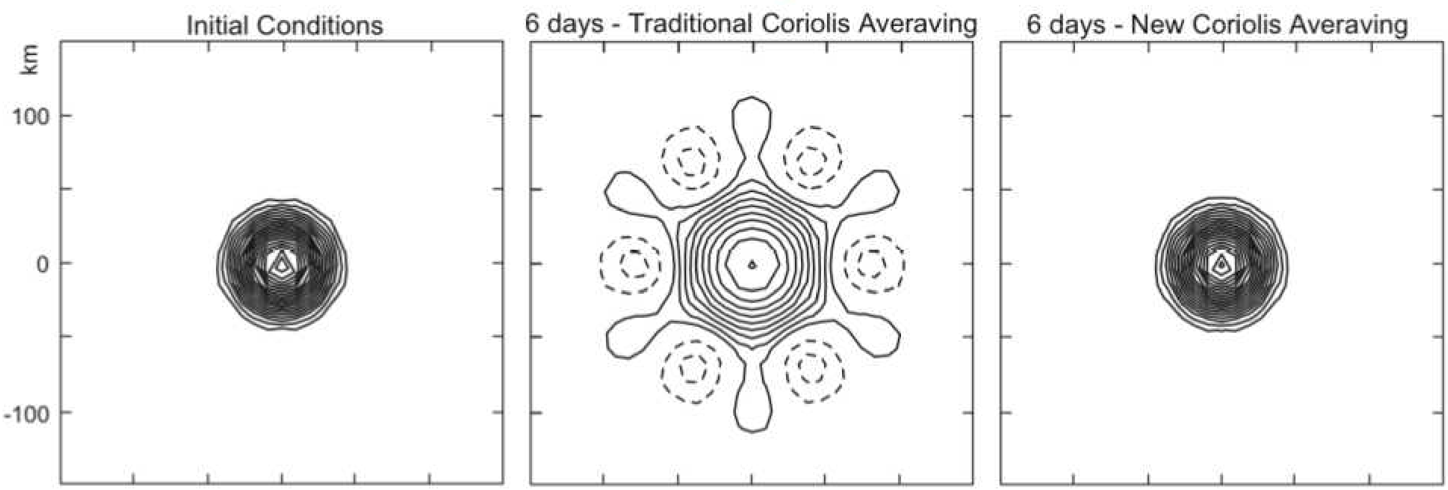
Nickovic (2002)

Équilibres géostrophiques préservés

Thuburn (2008)

Préservation discrète de l'équilibre géostrophique





Équilibres géostrophiques violés

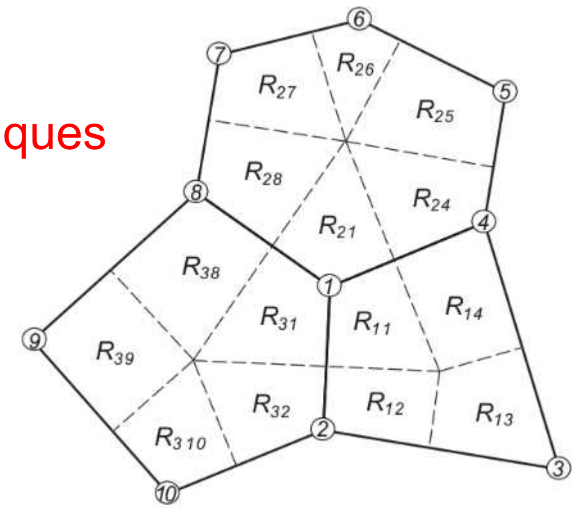
Nickovic (2002)

Équilibres géostrophiques préservés

Thuburn (2008)

Maillages quelconques :

Thuburn et al. (2009)



Vertical discretization :

- Vertical coordinates
- Where should we place prognostic/diagnostic variables
- Criteria : mass/transport consistency, numerical dispersion

Spherical meshes

- curvilinear Cartesian meshes
- global curvilinear Cartesian meshes : the pole problem
- a meshless method : spectral method
- quasi-uniform meshes and associated issues

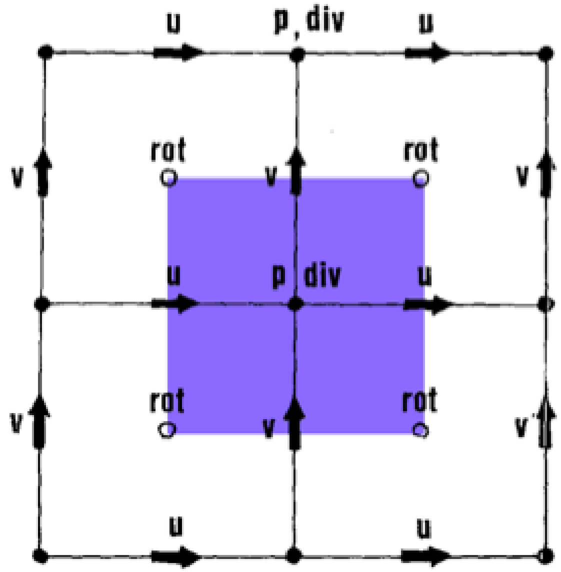
Conservation of non-linear integral invariants

- **a few clever solutions**
- **towards systematic approaches**

Energy-conserving schemes

- Arakawa (1966) : non-divergent, + enstrophy
- Sadourny et al. (1968) : non-divergent, icosahedron, + enstrophy
- Sadourny (1972) : rotating shallow-water, icosahedron/cubed sphere, collocated A-grid
- Sadourny (1975) : RSW, Cartesian staggered C-grid
- Simmons & Burridge (1981) : hydrostatic, hybrid vertical coordinate
- Janjic (1977, 1984), Rancic (1988, 2009) : RSW, staggered E-grid, cubed sphere / octagonal
- Arakawa & Lamb (1982) : RSW, C-grid, + enstrophy
- Hollingsworth et al. (1983) : RSW, Cartesian staggered C-grid
- Taylor et al. (2010) : RSW / hydrostatic, cubed-sphere, high-order "collocated" finite elements
- Ringler et al. (2010) : RSW, unstructured C- grid with orthogonal dual
- Thuburn & Cotter (2012) : RSW, unstructured C- grid with non-orthogonal dual
- Gassmann (2012) : $z-$based coordinate, non-hydrostatic, unstructured C-grid with orthogonal dual

Retro-engineering Sadourny (1975)



$$\partial_t h + \delta_x U + \delta_y V = 0$$

$$\partial_t u - \overline{q\overline{V}^{x}}^{y} + \delta_x B = 0$$

$$\partial_t v + \overline{q\overline{U}^{y}}^{x} + \delta_y B = 0$$

Expression for U,V,B to conserve a given energy H(h,u,v) ?

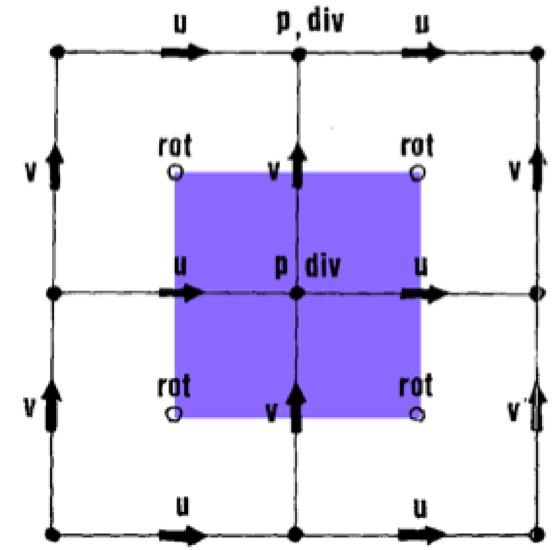
Sadourny, 1975

$$H = \frac{1}{2}\left\langle gh^2 + h\overline{u^2}^x + h\overline{v^2}^y \right\rangle$$

$$\partial_t h + \delta_x U + \delta_y V = 0 \qquad U = u\overline{h}^x = \frac{\partial H}{\partial u},$$

$$\partial_t u - \overline{q\overline{V}^x}^y + \delta_x B = 0 \qquad V = v\overline{h}^y = \frac{\partial H}{\partial v}$$

$$\partial_t v + \overline{q\overline{U}^y}^x + \delta_y B = 0 \qquad B = h + \frac{\overline{u^2}^x + \overline{v^2}^x}{2} = \frac{\partial H}{\partial h}$$



$$\frac{\mathrm{d}F}{\mathrm{d}t} = -\left\langle \frac{\partial F}{\partial u}\delta_x\frac{\partial H}{\partial h} - \frac{\partial H}{\partial u}\delta_x\frac{\partial F}{\partial h} + \frac{\partial F}{\partial v}\delta_y\frac{\partial H}{\partial h} - \frac{\partial H}{\partial v}\delta_y\frac{\partial F}{\partial h} + \overline{\frac{\partial F}{\partial v}}^x q\,\overline{\frac{\partial H}{\partial u}}^y - \overline{\frac{\partial F}{\partial u}}^y q\,\overline{\frac{\partial H}{\partial v}}^x \right\rangle$$

Hollingsworth et al., 1983 : Modified kinetic energy to control a numerical instability

$$H = \frac{1}{2}\left\langle gh^2 + \frac{h}{3}\left(\overline{u^2}^x + \overline{v^2}^y\right) + \frac{2h}{3}\left(\overline{u^2}^{xyy} + \overline{v^2}^{yxx}\right)\right\rangle$$

$$\Rightarrow U = \left(\frac{1}{3}\overline{h}^x + \frac{2}{3}\overline{h}^{xyy}\right)u = \frac{\partial H}{\partial u}, \qquad V = \left(\frac{1}{3}\overline{h}^y + \frac{2}{3}\overline{h}^{yxx}\right)v = \frac{\partial H}{\partial v}$$

Vector-invariant form and Poisson bracket

$$\frac{\partial h}{\partial t}+\nabla\cdot\frac{\delta H}{\delta \underline{v}}=0, \qquad \frac{\partial \underline{v}}{\partial t}+\frac{\mathrm{curl}\ \underline{v}}{h}\times\frac{\delta H}{\delta \underline{v}}+\nabla\frac{\delta H}{\delta h}=0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}F[h,\underline{v}]=\left\langle \frac{\delta H}{\delta \underline{v}}\cdot\nabla\frac{\delta F}{\delta h}-\frac{\delta F}{\delta \underline{v}}\cdot\nabla\frac{\delta H}{\delta h}+\frac{\mathrm{curl}\ \underline{v}}{h}\cdot\left(\frac{\delta F}{\delta \underline{v}}\times\frac{\delta H}{\delta \underline{v}}\right)\right\rangle=\{F,H\}$$

Recipe for an energy-conserving scheme

- discretize in vector-invariant form
- approximate total energy as a function of DOFs
- define mass flux and Bernoulli function from derivatives of total energy
- ensure that div and grad are *compatible*
- antisymmetrize the Coriolis bracket

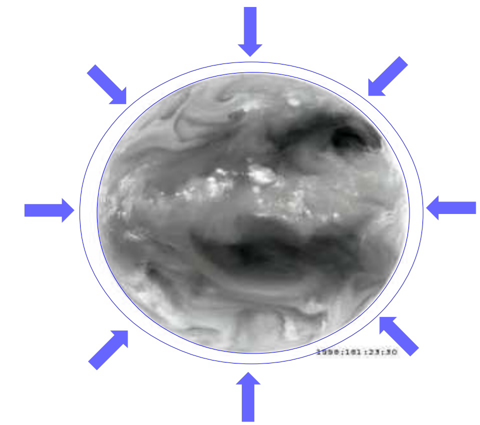
Same program in 3D ?

Eulerian vertical coordinate : Gassmann (2012)
Generalized vertical coordinate : Dubos & Tort (2014), Tort et al. (2014), Dubos at al. (2014)

Lagrangian least action principle for fluid flow
(Eckart, 1960 ; Morrison, 1998)

inertia Coriolis pressure gravity

$$\frac{D\dot{\mathbf{x}}}{Dt} + 2\boldsymbol{\Omega} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0$$



$$\delta \int \mathcal{L}\mathrm{d}t = 0$$

$$\Updownarrow$$

$$\frac{D}{Dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{1}{\rho}\nabla\left(\rho^2 \frac{\partial L}{\partial \rho}\right) + \frac{\partial L}{\partial \mathbf{x}}$$

$$L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s) = \frac{\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}}{2} + (\boldsymbol{\Omega} \times \mathbf{x}) \cdot \dot{\mathbf{x}} - gz - e\left(\frac{1}{\rho}, s\right)$$

$$\mathcal{L} = \mathcal{K} + \mathcal{C} - \mathcal{P} = \int L(\mathbf{x}, \dot{\mathbf{x}}, \rho, s)\mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2}\int \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}\mathrm{d}m$$

Kinetic energy

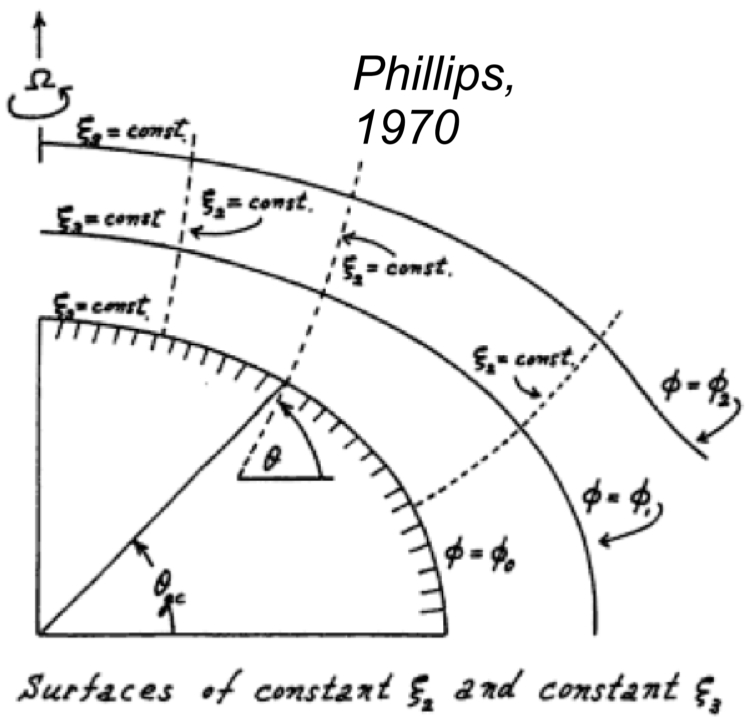
$$\mathcal{C} = \int (\boldsymbol{\Omega} \times \mathbf{x}) \cdot \dot{\mathbf{x}}\mathrm{d}m$$

Planetary velocity

$$\mathcal{P} = \int \left(e\left(\frac{1}{\rho}, s\right) + \Phi(\mathbf{x})\right)\mathrm{d}m$$

Internal energy
Potential energy

Dynamics in curvilinear coordinates



- *covariant* : same form in all coordinate systems
- derives from a *variational principle* : Hamilton's principle of least action
- *dynamically consistent* *(White & Bromley, 1995)* : conserves energy, angular momentum, potential vorticity for any choice of zonally-symmetric metric, planetary velocity, Jacobian

➡ metric, planetary velocity, Jacobian can be approximated without jeopardizing *dynamical consistency*

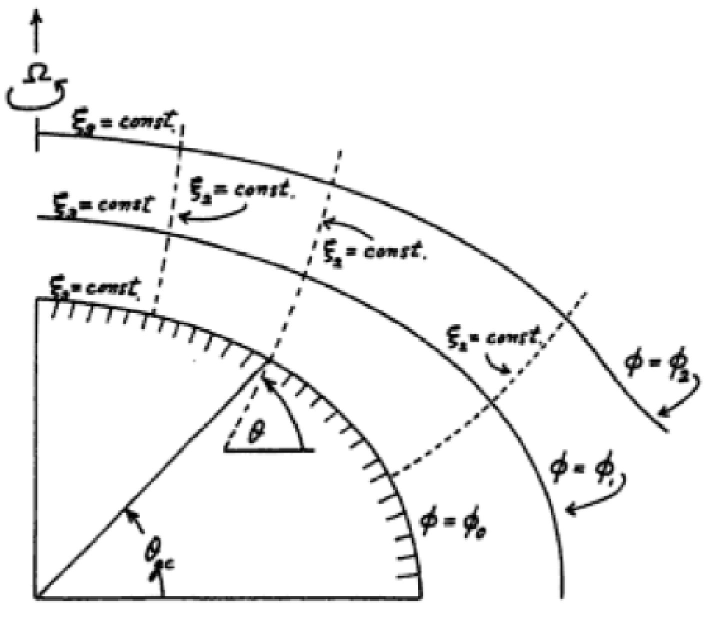
➡ various geometric approximations, each characterized by a certain choice of metric, planetary velocity, Jacobian

$$\frac{D\dot{\mathbf{x}}}{Dt} + \mathrm{curl}\ \mathbf{R} \times \dot{\mathbf{x}} + \frac{1}{\rho}\nabla p + \nabla \Phi = 0$$

⇕ *(Tort & Dubos, 2014)*

$$\begin{aligned} G_{ij}\frac{Du^j}{Dt} + \frac{1}{2}\left(\partial_j G_{ik} + \partial_k G_{ij} - \partial_i G_{jk}\right) u^j u^k \\ + \left[\partial_j R_i - \partial_i R_j\right] u^j + \frac{J}{\mu}\partial_i p + \partial_i \Phi &= 0 \end{aligned}$$

Least action principle in curvilinear coordinates



Surfaces of constant ξ_2 and constant ξ_3

$$(x,\, y,\, z) \rightarrow (\xi^1,\, \xi^2,\, \Phi)$$

$$\dot{x} \rightarrow u^i = \frac{D\xi^i}{Dt}$$

$$\dot{x} \rightarrow v_i = \frac{\partial L}{\partial u^i}$$

$$\boxed{\delta \int \mathcal{L} \mathrm{d}t = 0 \quad \Updownarrow \quad \frac{D}{Dt}\frac{\partial L}{\partial u^i} = \frac{1}{\hat{\rho}}\partial_i\left(\hat{\rho}^2 \frac{\partial L}{\partial \hat{\rho}}\right) + \frac{\partial L}{\partial \xi^i}}$$

$$\mathcal{L} = \int L\left(\xi^i, u^i, \hat{\rho}\right) \mathrm{d}m$$

$$\mathcal{K} = \frac{1}{2}\int G_{ij} u^i u^j \mathrm{d}m$$ Kinetic energy

$$\mathcal{C} = \int R_j u^j \mathrm{d}m$$ Planetary velocity

$$\mathcal{P} = \int \left(e\left(\frac{J}{\hat{\rho}},\, s\right) + \Phi(\xi^3)\right) \mathrm{d}m$$ Internal energy Potential energy

Generalized vertical coordinates & prognostic variables

Hydrostatic
Boussinesq
Anelastic

4+1 fields
Redundant flow description

3+1 fields
Non-redundant flow description

Euler

6 fields
Redundant flow description

5 fields
Non-redundant flow description

$\mu,\ s \text{ or } S = \mu s,\ \Phi$

\+ momentum

Lagrangian

$\mu,\ \Phi$

\+ momentum

Isentropic / Isopycnal

$\mu = \int M \mathrm{d}\eta,\ s \text{ or } S = \mu s,\ \Phi$

\+ momentum

Mass-based

$\mu,\ s \text{ or } S = \mu s$

\+ momentum

Eulerian (z-based)

Quasi-hydrostatic Hamiltonian formulation

(Dubos & Tort, 2014)

$$\begin{aligned}
\partial_t \mu + \partial_i \frac{\delta \mathcal{H}}{\delta v_i} + \partial_\eta \left(\mu \dot{\eta} \right) &= 0, \\
\partial_t \Theta + \partial_i \left(\theta \frac{\delta \mathcal{H}}{\delta v_i} \right) + \partial_\eta \left(\Theta \dot{\eta} \right) &= 0, \\
\partial_t v_i + \partial_\eta v_i \dot{\eta} + \frac{\partial_j v_i - \partial_i v_j}{\mu} \frac{\delta \mathcal{H}}{\delta v_j} + \partial_i \frac{\delta \mathcal{H}}{\delta \mu} + \theta \partial_i \left(\frac{\delta \mathcal{H}}{\delta \Theta} \right) &= 0,
\end{aligned}$$

4 equations of motion + 1 constraint

$$\frac{\delta \mathcal{H}}{\delta \Phi} = 0 \qquad \frac{\delta^2 \mathcal{H}}{\delta \Phi^2} \partial_t \Phi = r.h.s$$

Lagrangian / Isentropic / Mass-based

$$\frac{\delta^2 \mathcal{H}}{\delta \Phi^2} \left(\partial_\eta \Phi \, \dot{\eta} \right) = r.h.s$$

z-based

instantaneous hydrostatic adjustment

+ vertical remapping

Hamiltonian formulation in generalized vertical coordinates

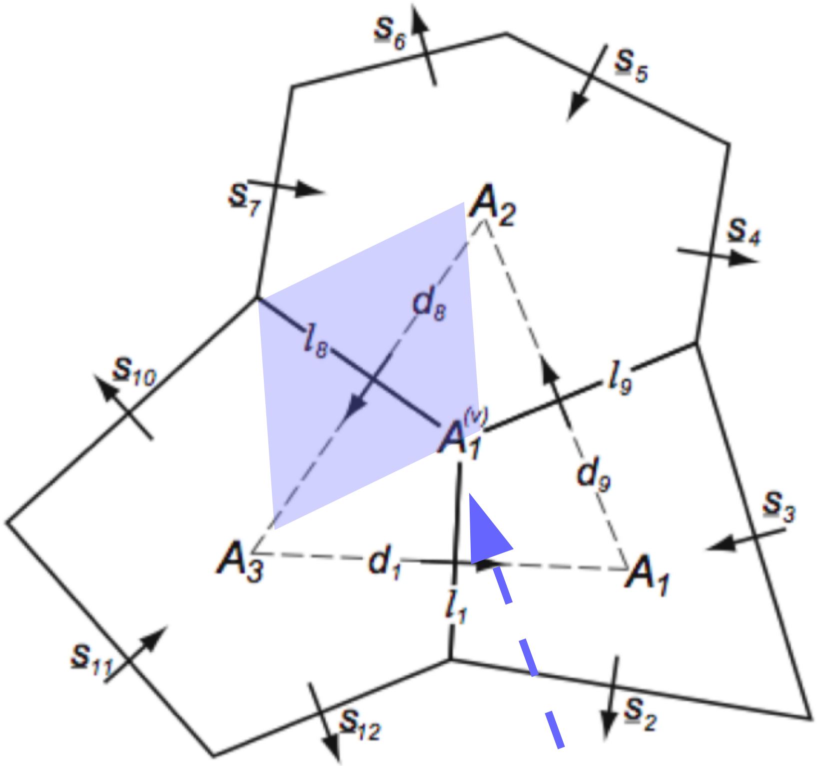
(Dubos & Tort, 2014)

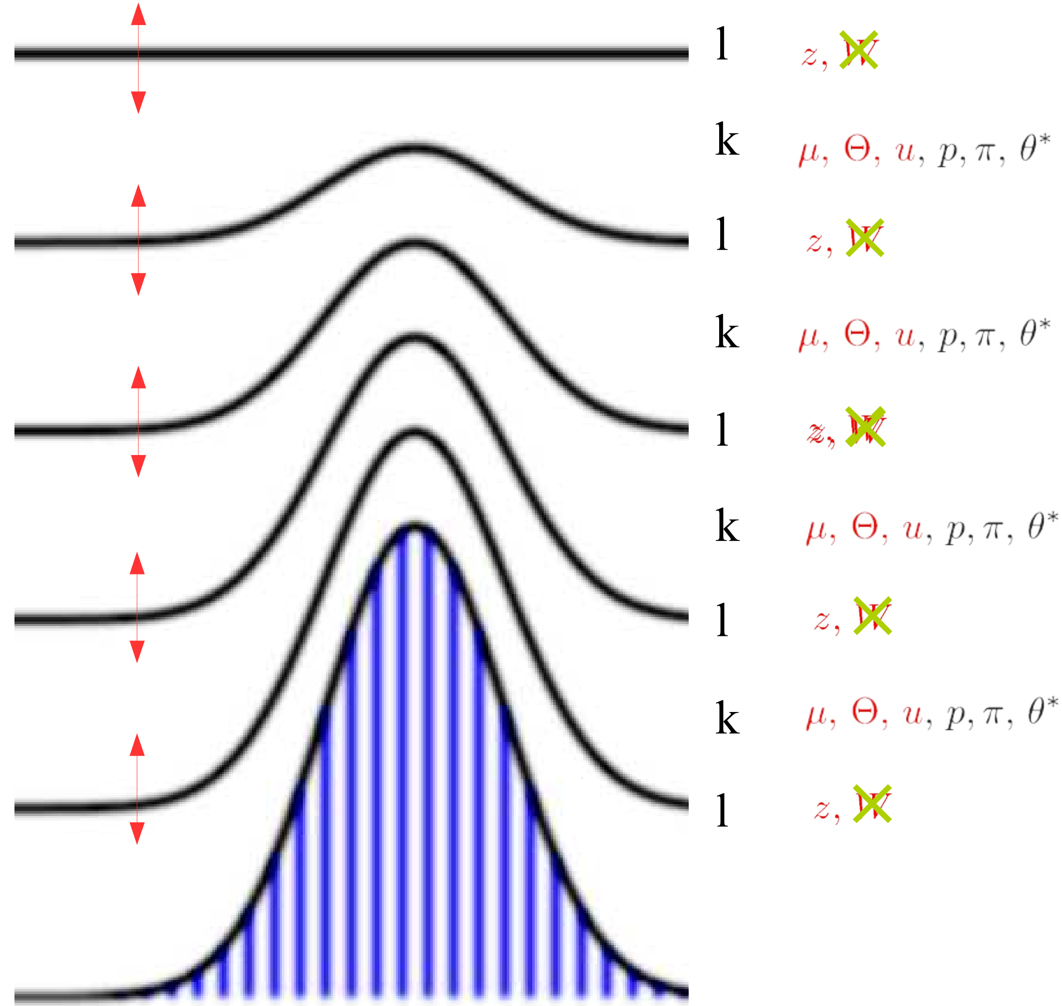
$$\mu\partial_\eta \frac{\delta\mathcal{H}}{\delta\mu} = (\partial_\eta v_i)\frac{\delta\mathcal{H}}{\delta v_i} - \partial_i\left(v_3\frac{\delta\mathcal{H}}{\delta v_i}\right) + (\partial_\eta\Phi)\frac{\delta\mathcal{H}}{\delta\Phi} - V_3\partial_\eta\frac{\delta\mathcal{H}}{\delta V_3} - \Theta\partial_\eta\frac{\delta\mathcal{H}}{\delta\Theta}$$

$$\begin{aligned}
\partial_t\mu + \partial_i\frac{\delta\mathcal{H}}{\delta v_i} + \partial_\eta(\mu\dot\eta) &= 0,\\
\partial_t\Theta + \partial_i\left(\theta\frac{\delta\mathcal{H}}{\delta v_i}\right) + \partial_\eta(\Theta\dot\eta) &= 0,\\
\partial_t v_i + (\partial_\eta v_i - \partial_i v_3)\,\dot\eta + \frac{\partial_j v_i - \partial_i v_j}{\mu}\frac{\delta\mathcal{H}}{\delta v_j} + \partial_i\left(\frac{\delta\mathcal{H}}{\delta\mu} + \dot\eta v_3\right) + \theta\partial_i\left(\frac{\delta\mathcal{H}}{\delta\Theta}\right) &= 0,\\
\partial_t V_3 + \partial_\eta(V_3\dot\eta) + \frac{\delta\mathcal{H}}{\delta\Phi} &= 0,\\
\partial_t\Phi + \dot\eta\partial_\eta\Phi - \frac{\delta\mathcal{H}}{\delta V_3} &= 0.
\end{aligned}$$

Integration by parts
+ independence to vertical coordinate
=> conservation of energy

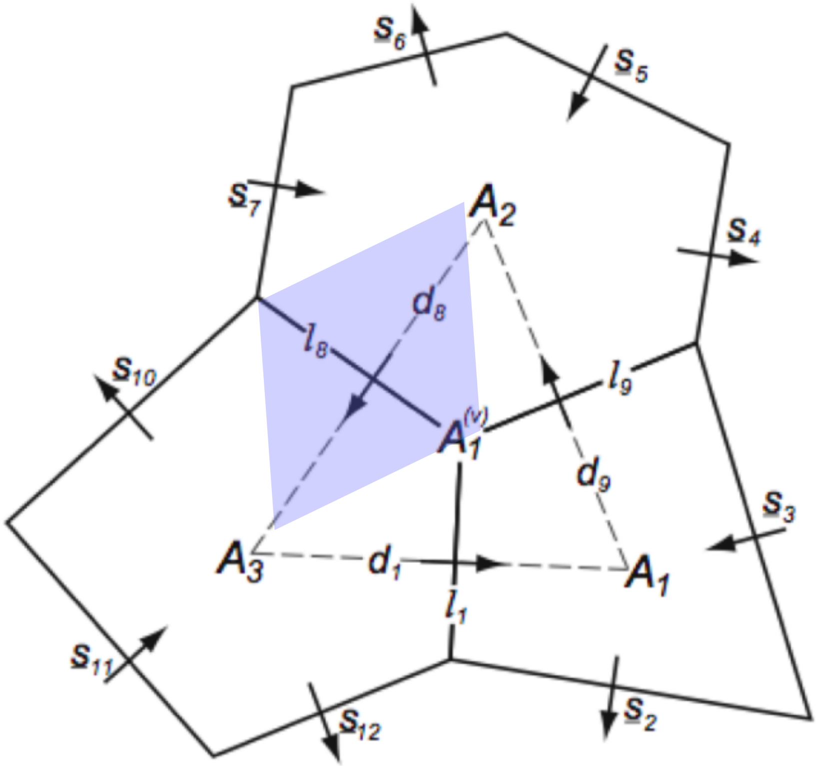
Isentropic / Isopycnal	*Mass-based*	*z-based*
$\dot\eta = 0$	Diagnosed from horizontal mass flux	$\partial_t\Phi = 0$

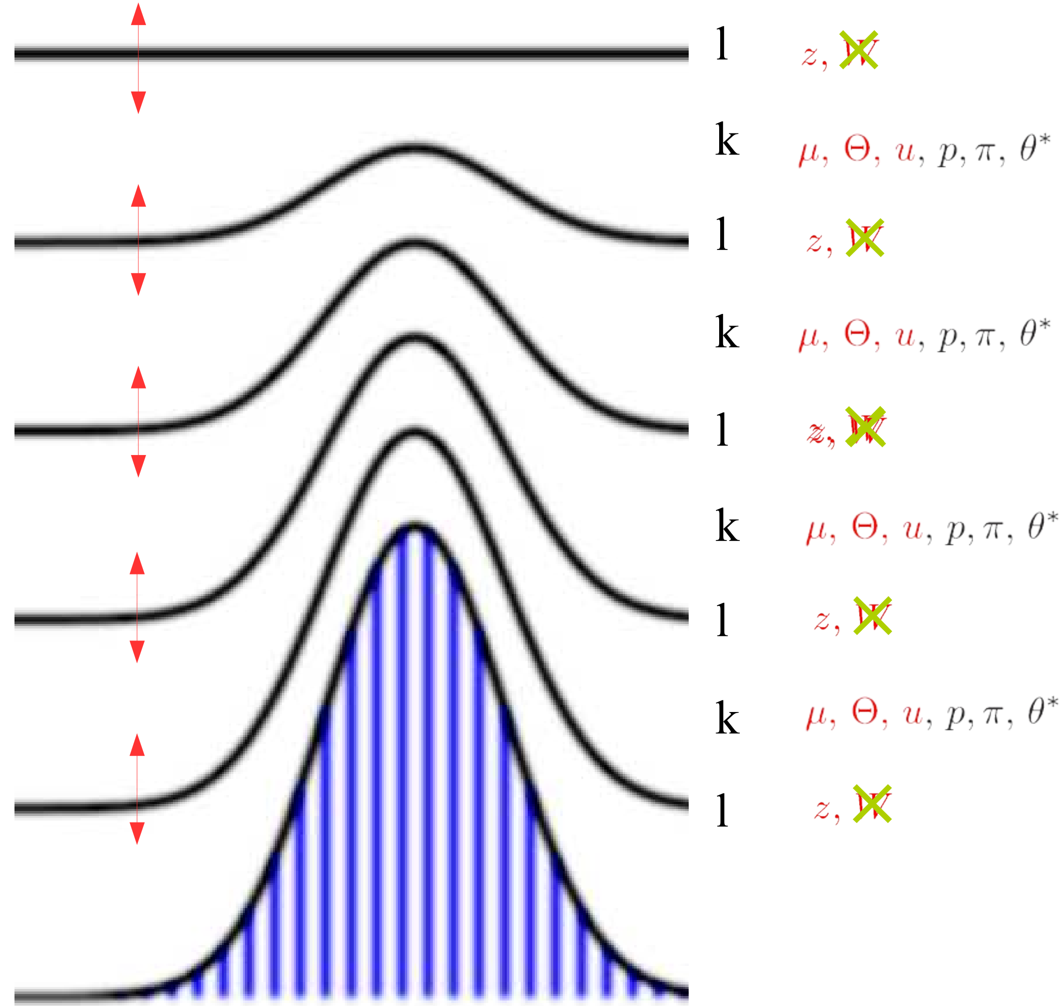




$$
\begin{aligned}
H &= K + P \\
K &= a^2 \sum_{ike} \mu_{ik} \frac{A_{ie}}{A_i} u_{ek}^2 \qquad \text{where } u_{ek} = \frac{v_{ek} - R_e}{a^2 d_e} \\
P &= \sum_{ik} \mu_{ik} \left(\overline{\Phi_i}^k + U\left(\frac{a^2 A_i \delta_k \Phi_i}{g \mu_{ik}}, \frac{\Theta_{ik}}{\mu_{ik}} \right) \right) + p_\infty a^2 g^{-1} \sum_i A_i \Phi_{iL}
\end{aligned}
$$

specific volume

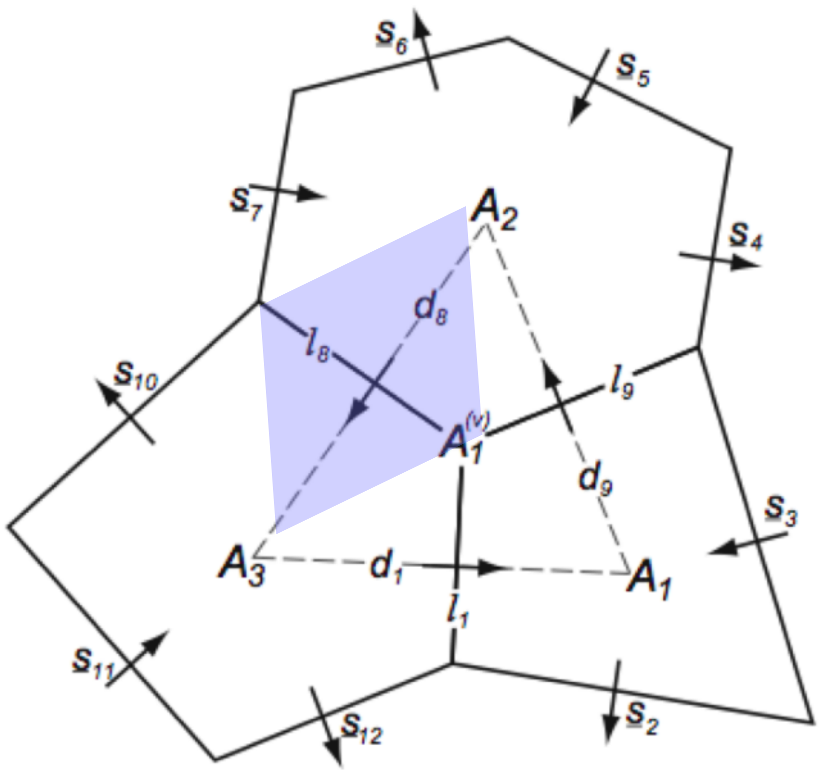




$$
\begin{aligned}
H' &= \sum_{ik}\left(\mu_{ik}\overline{\Phi'_i}^k - \frac{a^2 A_i \delta_k \Phi'_i}{g} p_{ik}\right) + p_\infty a^2 g^{-1} \sum_i A_i \Phi'_{iL} \\
&= \sum_{il}\left(\overline{\mu_i}^l + \frac{a^2 A_i}{g}\delta_l p_i\right)\Phi'_{il} + \sum_i \left(\frac{\mu_{iK}}{2} + \frac{a^2 A_i}{g}\left(p_\infty - p_{iK}\right)\right)\Phi'_{iL}
\end{aligned}
$$

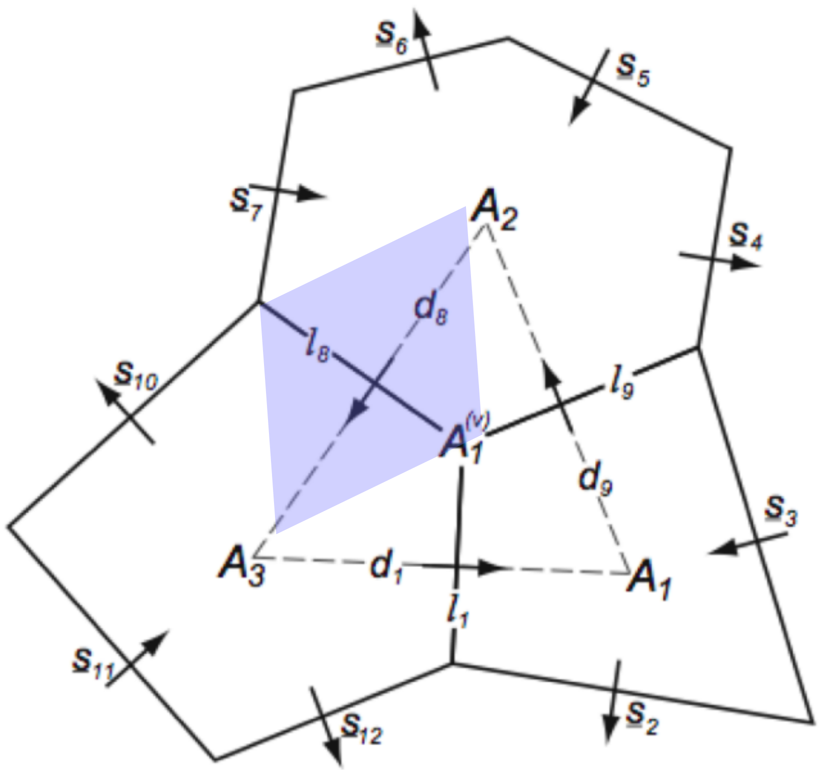
Hydrostatic balance

Top boundary condition



$$
\begin{aligned}
\pi_{ik} &= \frac{\partial H}{\partial \Theta_{ik}}, \\
U_{ek} &= \frac{\partial H}{\partial v_{ek}} = \overline{\left(\frac{\mu_k}{A}\right)}^{e} \frac{l_e}{d_e} u_{ek} \\
B_{ik} &= \frac{\partial H}{\partial \mu_{ik}} = a^2 \frac{\overline{l_e d_e u_e^2}^{i}}{A_i} + \overline{\Phi_i}^{k}
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \mu_{ik} + \delta_i\left(U_k\right) + \delta_k\left(W_i\right) &= 0 \\
\partial_t \Theta_{ik} + \delta_i\left(\theta_k^* U_k\right) + \delta_k\left(\theta_i^* W_i\right) &= 0 \\
\partial_t v_{ek} + \delta_e B_k + \theta_{ek}^* \delta_e \pi_k + \left(q_k U_k\right)_e^{\perp} + \overline{\left(\frac{\overline{W}^{k}}{\mu_k}\right)}^{e} \delta_l v_e^* &= 0
\end{aligned}
$$



Centered mass flux

$$U_{ek} = \frac{\partial H}{\partial v_{ek}} = \overline{\left(\frac{\mu_k}{A}\right)}^e \frac{l_e}{d_e} u_{ek}$$

$$\partial_t v_{ek} + \delta_e B_k + \theta^*_{ek} \delta_e \pi_k + (q_k U_k)_e^\perp + \overline{\left(\frac{\overline{W}^k}{\mu_k}\right)}^e \delta_l v_e^* = 0$$

$$(q_k U_k)_e^\perp = \sum_{e'} w_{ee'} \frac{q^*_{e'k} + q^*_{ek}}{2} U_{e'}$$

$$q_{vk} = \frac{\delta_v v_k}{\mu_v^*}$$

Energy and PV-conserving Coriolis force

Thuburn et al., 2009
Ringler et al., 2010

Vertical discretization :

- Vertical coordinates
- Where should we place prognostic/diagnostic variables
- Criteria : mass/transport consistency, numerical dispersion

Spherical meshes

- curvilinear Cartesian meshes
- global curvilinear Cartesian meshes : the pole problem
- a meshless method : spectral method
- quasi-uniform meshes and associated issues

Conservation of non-linear integral invariants

- a few clever solutions
- towards systematic approaches

References

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Thuburn & Woolings (2005)	J. Comput. Phys. **203** 386–404
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	NEMO	**ROMS**	**IFS/ARPEGE**	**MesoNH**	**WRF**	**EndGAME**	**LMDZ**	**DYNAMICO**
Geometry	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG+TSA	SG	SG+TSA	SG+TSA
Dynamics	HB	HB	FCE	A	FCE	FCE	HPE	HPE/(FCE)
Grid	CC	CC	LL	CC	CC	LL	LL	HEX
Disc. Dyn	FD	FV	SP	FD	FV	FD	FD	FD
Transport	FV	FV	SL	FV	FV	FV	FV	FV
Conserv.	M, E/Z	M		M	M	M	M, E/Z	M, E
Time	Split-EX	Split-EX	SI	EX	Split-HEVI	SI	EX	EX/HEVI
Helmholtz			Direct	Direct		Iter		

SG	Spherical-Geoid
TSA	Traditional Shallow-Atmosphere
FCE	Fully Compressible Euler
HPE	Hydrostatic Primitive Eq.
HB	Hydrostatic Boussinesq
A	Anelastic
EX	Explicit
SI	Semi-Implicit
Split	Split
HEVI	Horizontally Explicit, Vertically Implicit
Direct	Direct (spectral)
Iter	Iterative

CC	Cartesian Curvilnear
LL	Latitude-Longitude
HEX	Icosahedral-Hexagonal
FD	Finite Difference
FV	Finite Volume
FE	*Finite Element*
SE	*Spectral Element*
SL	Semi-Lagrangian
SP	Spectral
M	Mass and scalars
E	Energy
Z	Enstrophy